Professor Dr. Jens Vygen
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Exercise Set 2

Exercise 2.1. Let G be a bipartite graph with bipartition $V(G) = A \dot\cup B$, $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_k\}$. For any vector $x = (x_e)_{e \in E(G)}$ we define the matrix $M_G(x) = (m^x_{ij})_{1 \le i,j \le k}$ by

$$m^x_{ij} := \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Its determinant $\det M_G(x)$ is a polynomial in $x = (x_e)_{e \in E(G)}$. Prove that G has a perfect matching if and only if $\det M_G(x)$ is not identically zero.

(4 points)

Exercise 2.2. Let G be a bipartite graph.

(a) Let $V(G) = A \dot\cup B$ be a bipartition of G.
If $A' \subseteq A$ and $B' \subseteq B$, and there are a matching $M_{A'}$ covering A' and a matching $M_{B'}$ covering B', show that there must be a matching that covers $A' \cup B'$.

(b) Suppose that for every non-empty $E' \subseteq E(G)$ we have $\tau(G - E') < \tau(G)$. Show that $E(G)$ is a matching in G.

(3+1 points)

Exercise 2.3. Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Show the existence of an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \ge |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

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Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\texttt{min}\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\} > \texttt{min}\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(1+1+2+2+2+3+1=12 points)

Information: Submissions in groups of up to two students are allowed.

Deadline: October 25, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws18/coex.html`

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.