Exercise 1.1. Let $G$ be a graph and $M_1$ and $M_2$ be two inclusion-wise maximal matchings in $G$. Prove that $|M_1| \leq 2|M_2|$.

(4 points)

Exercise 1.2. Let $\alpha(G)$ denote the size of a maximum stable set in $G$, and $\zeta(G)$ the minimum cardinality of an edge cover. Prove:

(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph $G$,

(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph $G$ with no isolated vertices,

(c) $\zeta(G) = \alpha(G)$ for any bipartite graph $G$ with no isolated vertices.

(1 + 2 + 1 points)

Exercise 1.3. Let $G$ be a $k$-regular bipartite graph.

(a) Prove that $G$ contains $k$ disjoint perfect matchings.

Hint: Use König’s Theorem.

(b) Deduce from (a) that the edge set of any bipartite graph of maximum degree $k$ can be partitioned into $k$ matchings.

(2 + 3 points)

Exercise 1.4. Given a directed graph $G$, edge capacities $u : E(G) \to \mathbb{Z}_{\geq 0}$ and $s, t \in V(G)$, consider the linear programming formulation of the MAXIMUM FLOW PROBLEM:

$$\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^+(v)} x_e = \sum_{e \in \delta^-(v)} x_e \quad \text{for all } v \in V(G) \setminus \{s, t\} \\
& \quad x_e \leq u(e) \quad \text{for all } e \in E(G) \\
& \quad x_e \geq 0 \quad \text{for all } e \in E(G)
\end{align*}$$
Show that the dual LP always has an integral optimum solution, and deduce the Max-Flow Min-Cut Theorem from this.

Hint: Use the complementary slackness conditions.

(7 points)

Deadline: October 18, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ws18/coex.html

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.