Exercise 13.1: Let \( P \) the convex hull of the three points \((0, 0), (1, 0), \) and \((\frac{1}{2}, k)\) in \( \mathbb{R}^2 \), where \( k \in \mathbb{N} \). Prove that \( P^{(2k-1)} \neq P_I \), but \( P^{(2k)} = P_I \). (4 Points)

Exercise 13.2: Let \( P \subseteq [0, 1]^n \) be a polytope in the unit hypercube with \( P_I = \emptyset \). Prove that \( P^{(n)} = \emptyset \). (4 Points)

Exercise 13.3: Let \( S := \{ x \in \mathbb{Z}_+^2 : 4x_1 + x_2 \leq 28, x_1 + 4x_2 \leq 27, x_1 - x_2 \leq 1 \} \). Describe the facets of \( \text{conv}(S) \) by (iteratively) applying Gomory-Chvátal cuts. (A few cuts are sufficient, you may first find the facets geometrically) (4 Points)

Exercise 13.4: Let \( N := \{1, \ldots, n\} \) for \( n \in \mathbb{N} \) and \( c : N^2 \mapsto \mathbb{R} \). Consider the following ordering problem for computing a maximum-cost permutation \( \pi : N \to N \):

\[
\max \sum_{i,j \in N} c_{ij} x_{ij} \\
x_{ij} + x_{ji} = 1 \quad \forall 1 \leq i < j \leq n \\
x_{i_1i_2} + \cdots + x_{i_r i_1} \leq |C| - 1 \quad \forall C = \{j_1, \ldots, j_r\} \\
x \in \{0, 1\}^{n \times n}.
\]

Here \( x_{ij} = 1 \) for \( i \neq j \) can be interpreted as \( \pi(i) < \pi(j) \). The set of inequalities are called cycle inequalities.

1. Why does \( x \) determine a permutation?

2. How can the IP be used to compute a maximum-cost acyclic orientation of a complete graph with edge cost?

3. Prove that the cycle inequalities with \( |C| \geq 4 \) are redundant.

4. Prove that the cycle inequalities with \( |C| = 3 \) are facet-definint. (4 Points)

Submission deadline: Thursday, January 25, 2018, before the lecture (in groups of 2 students).