Winter semester 2017/18
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Linear and Integer Optimization

Exercise Sheet 11

Exercise 11.1: Let $\mathcal{F} = \{x \in \mathbb{Z}^n : Ax \leq b; x \geq 0\}$ with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Furthermore, let $F : \mathbb{R}^m \to \mathbb{R}$ be a function that is *superadditive*, i.e. $F(a_1) + F(a_1) \leq F(a_1 + a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$, *non-decreasing*, i.e. $F(a_1) \leq F(a_2)$ for $a_1 \leq a_2$, and that fulfills $F(0) = 0$.

1. Prove that the inequality
$$\sum_{j=1}^{n} F(A_j)x_j \leq F(b)$$
holds for all $x \in \text{conv}(\mathcal{F})$, where A_j is the j-th column of A. (3 Points)

2. Conclude, that the following inequalities hold for all $x \in \text{conv}(\mathcal{F})$:
$$\sum_{j=1}^{n} \lfloor u^\intercal A_j \rfloor x_j \leq \lfloor u^\intercal b \rfloor$$
for all $u \in \mathbb{R}^m_{\geq 0}$. (1 Point)

Exercise 11.2: An **unimodular matrix** is an integral square matrix A with $\det(A) \in \{-1, 1\}$. Prove that any unimodular matrix arises from the identity matrix by elementary unimodular column operations. (4 Points)

Exercise 11.3: Prove the integral version of Carathéodory's theorem. For any pointed rational polyhedral cone $C \subset \mathbb{R}^n$, any Hilbert basis $\{a_1, \ldots, a_l\}$ of C and any integral point $x \in C \cap \mathbb{Z}^n$, there are $2n - 1$ vectors from the Hilbert basis such that x is a non-negative integral combination of these vectors. (4 Points)

Exercise 11.4: Consider the following capacitated facility location problem: given a set of clients C and a set of potential facility locations F, a metric ℓ on $C \cup F$ representing connection costs, facility opening costs $p : F \to \mathbb{R}_{\geq 0}$ and capacities $c : F \to \mathbb{N}$, and client demands $d : C \to \mathbb{N}$, find a set $I \subseteq F$ of facilities to be opened and an assignment $f : C \to I$ of clients to open facilities such that the capacity bounds are respected ($\sum_{c \in f^{-1}(x)} d(c) \leq c(x)$ for all $x \in I$) and the sum of opening costs and connection costs of clients to their assigned facilities is minimized.

1. Model this problem as an integer program. (2 Points)
2. Give a non-trivial example instance for which the LP relaxation of your IP has a unique optimum solution which is integral. Give an example instance for which every optimal solution of the LP relaxation is fractional. (2 Points)

Submission deadline: Thursday, Jaunary 11, 2018, before the lecture (in groups of 2 students).

We wish all participants a merry christmas (or holidyas if you prefer) and a happy new year 2018!