Winter semester 2017/18
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Linear and Integer Optimization

Exercise Sheet 7

Exercise 7.1: Consider a minimum cost flow problem (G, u, b, c) with $u \in \mathbb{N}^{E(G)}$, $b \in \mathbb{Z}^{V(G)}$, and a fixed order $V(G) = \{v_1, v_2, \ldots, v_n\}$ of the vertices. Define a balance perturbation $\epsilon \in \mathbb{K}^{V(G)}$ by $\epsilon(v_i) = \frac{1}{n}$ for $i = 2, 3, \ldots, n$, $\epsilon(v_1) = -\frac{n-1}{n}$, and consider the modified problem $(G, y, b + \epsilon, c)$.

1. Let (r, T) be a spanning tree solution with $r = v_1$. Let $D(v)$ be the set of vertices $v' \in V(G)$ whose v'-r-path through the tree contains v. Note $v \in D(v)$.

   Show that the perturbation $b + \epsilon$ decreases the flow of each downward arc (v, w) by $|D(w)|/n$, and increases the flow of each upward arc (v, w) by $|D(v)|/n$.

   Furthermore, show that the flow value on a tree edge of a feasible spanning tree structure for $(G, u, b + \epsilon, c)$ is non-zero and a multiple of $\frac{1}{n}$. (3 Points)

2. Use part one to show that the network simplex algorithm solves the perturbed problem in pseudopolynomial time, independent of the choice of the entering edge.

   (1 Point)

3. Show that a spanning tree structure (r, T, L, U) is strongly feasible for (G, u, b, c) if and only if it is strongly feasible for $(G, u, b + \epsilon, c)$. Conclude that the network simplex algorithm has a pseudopolynomial running time, when generating strongly feasible spanning tree solutions, independent of the choice of the entering edge.

   (2 Points)

Exercise 7.2:

1. Let $A \in \mathbb{Q}^{m \times n}$ and $B \in \mathbb{Q}^{n \times p}$ be two matrices. Prove:

$$size(AB) \leq 2(p \cdot size(A) + m \cdot size(B)).$$

   (2 Points)

2. Let $A \in \mathbb{Q}^{n \times n}$ be an invertible matrix. Prove:

$$size(A^{-1}) \leq 4n^2 \cdot size(A).$$

   (3 Points)

Exercise 7.3: Let $(F_n)_{n\in\mathbb{N}}$ be the sequence of Fibonacci numbers, i.e. $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove:

1. $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. (2 Points)
2. In each iteration i of the continuous fraction expansion, we have $h_i \geq F_{i+1}$. (1 Point)
3. The continuous fraction expansion with input $\frac{p}{q}$ terminates after $O(\log q)$ iterations. (2 Points)

Submission deadline: Thursday, November 30, 2017, before the lecture (in groups of 2 students).