Exercise 4.1: Let \( P = P(A, b) \) be a polyhedron and \( F \) a minimal face of \( P \). Prove that \( Ax = Ay \) holds for all \( x, y \in F \). (3 Points)

Exercise 4.2: Let \( C \) be a convex cone and \( -C \) the cone \( \{ x : -x \in C \} \). We call \( L = (C \cap -C) \) the **lineality space** of \( C \).

a) Prove that \( \bar{C} := C \cap L^\perp \), where \( L^\perp = \{ u : u^\top x = 0 \ \forall x \in L \} \), is a pointed cone and that \( C \) is the direct sum of its lineality space \( L \) and the pointed cone \( \bar{C} \), i.e.

\[
C = (C \cap L^\perp) \oplus L.
\]

(2 Points)

b) Show that each polyhedron \( P \) has a decomposition \( P = (Q + C) \oplus L \), where \( Q \) is a polytope, \( C \) a pointed cone and \( L \) a linear subspace. (3 Points)

Exercise 4.3: Given two extreme points \( a \) and \( b \) of a polyhedron \( P \), we say that they are **adjacent** on \( P \) if the line segment between them forms an edge (i.e. a face of dimension 1) of \( P \). Prove that \( a \) and \( b \) are adjacent on \( P \) if and only if there exists a cost function \( c \) such that \( a \) and \( b \) are the only two extreme points of \( P \) minimizing \( c^\top x \) over \( P \). (3 Points)

Exercise 4.4: Let \( H = (V, E) \) be a hypergraph, i.e. \( V \) is a finite set of vertices and \( E \subseteq 2^V \). Furthermore, let \( F \subseteq V \) and \( x, y : F \to \mathbb{R} \). Provide an LP formulation for the following problem and dualize the LP:

Determine (an extension) \( x, y : V \setminus F \to \mathbb{R} \) such that

\[
\sum_{v \in E} (\max_{v \in e} x(v) - \min_{v \in e} x(v)) + \max_{v \in e} y(v) - \min_{v \in e} y(v)
\]

is minimized. (5 Points)
Remark: This is a relaxation of the placement problem in chip design. The vertices correspond to connected modules that must be placed minimizing the length of all interconnects (hyperedges). Vertices in $F$ are preplaced. The problem becomes much harder when requiring disjointness of the modules.

Submission deadline: Thursday, November 9, 2017, before the lecture (in groups of 2 students).