Exercise 2.1: Let $P$ and $Q$ be two polyhedra in $\mathbb{K}^n$. Consider the convex hull of their union $\text{conv}(P \cup Q)$ and prove that its closure $\text{conv}(P \cup Q)$ is a polyhedron. Furthermore, specify polyhedra $P, Q$ such that $\text{conv}(P \cup Q)$ is not a polyhedron.

The convex hull of a set $Y \subseteq \mathbb{K}^n$ is defined as $\text{conv}(Y) := \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, 0 \leq \lambda_i \leq 1, x_i \in Y \text{ for } 1 \leq i \leq k; \sum_{i=1}^{k} \lambda_i = 1 \right\}$. (4 Points)

Exercise 2.2: (Three characterizations of vertices)

Let $A \in \mathbb{K}^{m \times n}$, $b \in \mathbb{K}^m$, and $P = P(A, b) := \{ x \in \mathbb{K}^n : Ax \leq b \}$. An element $x \in P$ is called an extreme point if there are no two elements $y, z \in P \setminus \{ x \}$ such that $x$ is a convex combination of $y$ and $z$, i.e. there is no $\lambda \in [0, 1]$ with $x = \lambda y + (1 - \lambda) z$.

1. If $x^*$ is a vertex of $P$, then $x^*$ is also an extreme point of $P$. (2 Points)

2. If there is a subsystem $A'x \leq b'$ of $Ax \leq b$ for which $A'x^* = b'$ with $\text{rank}(A') = n$, then $x^*$ is a vertex of $P$. (3 Points)

3. Let $x^*$ be an extreme point of $P$, then there is a subsystem $A'x \leq b'$ of $Ax \leq b$ for which $A'x^* = b'$ and $\text{rank}(A') = n$ hold. (3 Points)

Exercise 2.3: Prove that any set $X \subseteq \mathbb{K}^n$ with $|X| > n + 1$ can be decomposed into subsets $X_1$ and $X_2$ such that $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$. (4 Points)

Exercise 2.4: Let $X \subseteq \mathbb{K}^n$ and $y \in \text{conv}(X)$. Prove that there exist $x_1, \ldots, x_{n+1} \in X$ with $y \in \text{conv}(\{x_1, \ldots, x_{n+1}\})$. (4 Points)

Submission deadline: Thursday, October 26, 2017, before the lecture (in groups of 2 students).

Programming Exercise on the back!
Programming Exercise 1

Implement the Fourier-Motzkin elimination algorithm to decide if an LP $\max \{c^\top x : Ax \leq b\}$ has a feasible solution. If it has a solution, print a solution vector to the standard output as a single line. If it does not have a solution, print the string “empty” followed by a certificate vector $y$ according to Farkas Lemma (in one line). The program has to be implemented in C/C++ and compile with either the GNU compilers gcc/g++ or clang/clang++. The program should be run from the command line and read in a text file, whose name is given as an argument. The text file specifies the LP in the following format.

- The first line contains the number $m$ of rows and $n$ of columns of $A$.
- The second line contains $n$ floating point numbers specifying $c$.
- The third line contains $m$ floating point numbers specifying $b$.
- The next $m$ lines contain the rows of $A$. Each line contains the $n$ floating point numbers in the respective row.

On the web site to the exercises you find test instances and an example program in C for reading the input. You may use the example as a base for your implementation.

(20 Points)

Submission of the programming exercise until Thursday, November 9, 2017, before the lecture via e-mail to your tutor and held@or.uni-bonn.de!