Winter semester 2017/18 Research Institute for Discrete Mathematics
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Linear and Integer Optimization

Exercise Sheet 2

Exercise 2.1: Let P and Q be two polyhedra in $\mathbb{K}^n$. Consider the convex hull of their union $conv(P \cup Q)$ and prove that its closure $\overline{conv(P \cup Q)}$ is a polyhedron. Furthermore, specify polyhedra P, Q such that $conv(P \cup Q)$ is not a polyhedron.
The convex hull of a set $Y \subseteq \mathbb{K}^n$ is defined as $conv(Y) := \left\{\sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, 0 \leq \lambda_i \leq 1, x_i \in Y \text{ for } 1 \leq i \leq k; \sum_{i=1}^{k} \lambda_i = 1\right\}$. (4 Points)

Exercise 2.2: (Three characterizations of vertices)
Let $A \in \mathbb{K}^{m\times n}$, $b \in \mathbb{K}^m$, and $P = P(A, b) := \{x \in \mathbb{K}^n \ : \ Ax \leq b\}$. An element $x \in P$ is called an **extreme point** if there are no two elements $y, z \in P \setminus \{x\}$ such that x is a convex combination of y and z, i.e. there is no $\lambda \in [0, 1]$ with $x = \lambda y + (1 - \lambda)z$.
Let $x^\star \in P$. Show

1. If $x^\star$ is a vertex of P, then $x^\star$ is also an extreme point of P. (2 Points)
2. If there is a subsystem $A'x \leq b'$ of $Ax \leq b$ for which $A'x^\star = b'$ with $rank(A') = n$, then $x^\star$ is a vertex of P. (3 Points)
3. Let $x^\star$ be an extreme point of P, then there is a subsystem $A'x \leq b'$ of $Ax \leq b$ for which $A'x^\star = b'$ and $rank(A') = n$ hold. (3 Points)

Exercise 2.3: Prove that any set $X \subseteq \mathbb{K}^n$ with $|X| > n + 1$ can be decomposed into subsets X_1 and X_2 such that $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$. (4 Points)

Exercise 2.4: Let $X \subseteq \mathbb{K}^n$ and $y \in \text{conv}(X)$. Prove that there exist $x_1, \ldots, x_{n+1} \in X$ with $y \in \text{conv}(\{x_1, \ldots, x_{n+1}\})$.
(4 Points)

Submission deadline: Thursday, October 26, 2017, before the lecture (in groups of 2 students).

Programming Exercise on the back!

Programming Exercise 1
Implement the Fourier-Motzkin elimination algorithm to decide if an LP $\max\{c^\intercal x : Ax \leq b\}$ has a feasible solution. If it has a solution, print a solution vector to the standard output as a single line. If it does not have a solution, print the string "`empty`" followed by a certificate vector y according to Farkas Lemma (in one line). The program has to be implemented in C/C++ and compile with either the GNU compilers `gcc/g++` or `clang/clang++`. The program should be run from the command line and read in a text file, whose name is given as an argument. The text file specifies the LP in the following format.

- The first line contains the number m of rows and n of columns of A.
- The second line contains n floating point numbers specifying c.
- The third line contains m floating point numbers specifying b.
- The next m lines contain the rows of A. Each line contains the n floating point numbers in the respective row.

On the web site to the exercises you find test instances and an example program in C for reading the input. You may use the example as a base for your implementation.
(20 Points)

Submission of the programming exercise until Thursday, November 9, 2017, before the lecture via e-mail to your tutor **and** to held@or.uni-bonn.de!