Exercise Set 9

Exercise 9.1. Let $G$ be a simple undirected graph such that $|\delta(v)|$ is odd for every $v \in V(G)$. For an edge $e \in E(G)$, let $\mathcal{H}_e$ denote the family of all Hamiltonian cycles in $G$ containing $e$. Show that $|\mathcal{H}_e|$ is even for every $e \in E(G)$. (4 points)

Exercise 9.2. Let $n \geq 3$ and let $f : V(K_n) \to \mathbb{R}^2$ be an injective function. We define a function $g_f : E(K_n) \to \mathcal{P}(\mathbb{R}^2)$ by mapping an edge $\{u,v\}$ to the open line-segment between $f(u)$ and $f(v)$ (i.e. the line-segment between $f(u)$ and $f(v)$ without its two endpoints). We say that two edges $e,e' \in E(K_n)$ cross (with respect to $f$) if $g_f(e) \cap g_f(e') \neq \emptyset$.

Given a Hamiltonian tour $T$ in $K_n$ and two edges $e,e' \in E(T)$ which cross, consider the algorithm REMOVECELLING($T,e,e'$): Let $u,v$ be the endpoints of $e$ and $u',v'$ be the endpoints of $e'$. Delete $e$ and $e'$ from $T$. If $T + \{u,u'\} + \{v,v'\}$ is a Hamiltonian tour, add $\{u,u'\}$ and $\{v,v'\}$ to $T$. Otherwise add $\{u,v'\}$ and $\{v,u'\}$ to $T$.

(i) Show that after REMOVECELLING($T,e,e'$), $T$ is still a Hamiltonian tour.

Consider the algorithm REMOVEALLCELLINGS($T$): While there are edges in $T$ that cross, choose one such pair of edges $e,e' \in E(T)$ and call REMOVECELLING($T,e,e'$).

(ii) Assuming that in the range of $f$ no three points are colinear (i.e. lie on the same straight-line), show that for any Hamiltonian tour $T$ the algorithm REMOVEALLCELLINGS($T$) terminates after $O(n^3)$ calls of REMOVECELLING.

(iii) Give an example (i.e. a choice of $n$, $f$ and $T$) such that REMOVEALLCELLINGS($T$) does not terminate. (1+4+1 points)
**Exercise 9.3.** Let \( n \geq 3 \) and consider the complete graph \( K_n \) with an edge-weight function \( c : E(K_n) \to \mathbb{R}_{\geq 0} \). Given a (fixed) partition \((X_1, \ldots, X_k)\) of \( V(K_n) \), we say that \( e \in E(K_n) \) is an *intra-part edge* if \( e \in E(K_n[X_1]) \cup \ldots \cup E(K_n[X_k]) \). Otherwise, i.e. if \( e \in \delta(X_1) \cup \ldots \cup \delta(X_k) \) we say \( e \) is an *inter-part edge*.

Let \((X_1, \ldots, X_k)\) be a partition of \( V(K_n) \) such that every intra-part edge has weight 0.

(i) Show that every optimal solution of the TSP for \((K_n, c)\) uses at most \( k(k-1) \) edges of strictly positive weight.

(ii) Show that there is at least one optimal solution of the TSP for \((K_n, c)\) which uses at most \( k(k-1) \) inter-part edges.

(iii) Show that, using the partition \((X_1, \ldots, X_k)\), we can find an optimal solution of the TSP for \((K_n, c)\) in \( O(n^{2k(k-1)+1}) \)-time.

(1+1+4 points)

**Deadline:** December 14th, before the lecture. The websites for lecture and exercises can be found at:

[http://www.or.uni-bonn.de/lectures/ws17/co_exercises/exercises.html](http://www.or.uni-bonn.de/lectures/ws17/co_exercises/exercises.html)

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