Exercise Set 8

Exercise 8.1. Let \( G \) be a graph. A 2-cover of \( G \) is a function \( y : V(G) \to \{0, 1, 2\} \) with \( y(v) + y(w) \geq 2 \) for all \( \{v, w\} \in E(G) \). The size of \( y \) is \( \sum_{v \in V(G)} y(v) \).

If \( y \) is a 2-cover, the set \( \{v \in V(G) : y(v) = 0\} \) is a stable set.

Conversely, a stable set \( A \subseteq V(G) \) determines a 2-cover \( y \) by setting

\[
y(v) = \begin{cases} 
0 & \text{if } v \in A, \\
2 & \text{if } v \in N(A), \\
1 & \text{otherwise}.
\end{cases}
\]

Prove:

(i) The maximum size of a 2-matching in \( G \) equals the minimum size of a 2-cover of \( G \), where the size of a 2-matching \( f : E(G) \to \{0, 1, 2\} \) is \( \sum_{e \in E(G)} f(e) \).

(ii) \( G \) has a perfect 2-matching iff \( |N(A)| \geq |A| \) for all stable sets \( A \subseteq V(G) \).

(4 points)

Exercise 8.2. Let \( G \) be a graph, \( b : V(G) \to \mathbb{N} \), and \( c : E(G) \to \mathbb{R} \) a weight function.

(i) Show that the uncapacitated maximum-weight \( b \)-matching problem in bipartite graphs can be solved in strongly polynomial time.

(ii) Use Step (i) to show that the uncapacitated maximum-weight \( b \)-matching problem can be solved in strongly polynomial time if \( b \) is even.

(iii) Use Step (ii) to show that the uncapacitated maximum-weight \( b \)-matching problem can be solved in strongly polynomial time.

(iv) Use Step (iii) to show that the capacitated maximum-weight \( b \)-matching problem for edge capacities \( u : E(G) \to \mathbb{N} \cup \{\infty\} \) can be solved in strongly polynomial time.

(8 points)
Exercise 8.3. Let $G$ be an undirected graph and $T \subseteq V(G)$ with $|T| = 2k$ even. Prove that the minimum cardinality of a $T$-cut in $G$ equals the maximum of $\min_{i=1}^{k} \lambda_{s_i,t_i}$ over all pairings $T = \{s_1, t_1, \ldots, s_k, t_k\}$, where $\lambda_{s,t}$ denotes the maximum number of pairwise edge-disjoint $s$-$t$-paths.

(4 points)

Deadline: December 7th, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ws17/co_exercises/exercises.html

In case of any questions feel free to contact me at silvanus@or.uni-bonn.de.