Exercise Set 2

Exercise 2.1. Let $G_0$ be a (not necessarily simple) bipartite graph with bipartition $A \cup B$. Assume that $|N_{G_0}(X)| \geq |X| + 1$ for all $\emptyset \neq X \subseteq A$, and for every $b \in B$, let $G_b$ be a factor-critical graph. Let $H$ be a graph that contains a perfect matching. Define a graph $G$ on the vertex set $V(G) = A \cup V(H) \cup \bigcup_{b \in B} V(G_b)$ as follows. Keep all edges of the graphs $H$ and $G_b$ for $b \in B$. For every edge $\{a, b\} \in E(G_0)$ with $a \in A$, $b \in B$, add an edge connecting $a$ to an arbitrary vertex of $G_b$. Add edges connecting vertices in $A$ with vertices in $V(H)$ arbitrarily, and add arbitrary edges connecting elements of $A$. Let $X, Y, W$ denote the Gallai-Edmonds decomposition of $G$. Show that $X = A$, $Y = \bigcup_{b \in B} V(G_b)$ and $W = V(H)$. Show that every graph arises by the above construction uniquely.

(4 points)

Exercise 2.2. Let $G$ be a graph and $M$ a matching in $G$ that is not maximum. In this exercise we use the terminology disjoint subgraphs/paths/circuits and mean it quite literally: Two subgraphs are disjoint if they have no edges and no vertices in common. (Note that the term vertex-disjoint paths is often used to mean that two paths have no inner-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint $M$-augmenting paths in $G$.

(ii) Show the existence of an $M$-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let $P$ be a shortest $M$-augmenting path in $G$ and $P'$ an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then $P_i$ and $P_j$ are disjoint.
(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let $G$ be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching $M$ in $G$ show that we can find in $O(n + m)$-time a family $\mathcal{P}$ of disjoint shortest $M$-augmenting paths such that if $M'$ is the matching obtained by augmenting $M$ over every path in $\mathcal{P}$, then

$$\min\{|E(P)| : P \text{ is an } M'-\text{augmenting path}\} > \min\{|E(P)| : P \text{ is an } M-\text{augmenting path}\}$$

(vii) Describe an algorithm with runtime $O(\sqrt{\pi(m + n)})$ that solves the \textbf{Cardinality Matching Problem} in bipartite graphs.

(12 points)

\textbf{Deadline:} October 26\textsuperscript{th}, before the lecture. The websites for lecture and exercises can be found at

\url{http://www.or.uni-bonn.de/lectures/wsi7/coex.html}

In case of any questions feel free to contact me at silvanus@or.uni-bonn.de.