Winter term 2016/17
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Combinatorial Optimization
Exercise set 10

Exercise 10.1: Let G be an undirected graph. Given a partition $(X_1, \dots, X_k)$ of $V(G)$ we define $\delta(X_1, \dots, X_k) := \delta(X_1) \cup \dots \cup \delta(X_k)$ (so, in particular, if $\emptyset \neq X \subsetneq V(G)$ we have $\delta(X) = \delta(X, V(G) \setminus X)$). Consider the polytope

$$R_G := \left\{ x : E(G) \to [0,1] : \sum_{e \in E(G)} x(e) = |V(G)| - 1 \text{ and } \sum_{e \in \delta(X_1,\dots,X_k)} x(e) \geq k - 1 \text{ for every partition } (X_1, \dots, X_k) \text{ of } V(G) \right\}$$

(Compare this with exercise 7.4.) Show that R_G is the spanning-tree polytope of G. (4 points)

Exercise 10.2: Consider Nearest Neighbor Heuristic: Given an instance (K_n, c) of the TSP with $V(K_n) = \{1, \dots, n\}$, let $v_1 := 1$. For $i = 2, \dots, n$: Choose $v_i \in V(K_n) \setminus \{v_1, \dots, v_{i-1}\}$ such that $c(\{v_{i-1}, v_i\})$ is smallest possible. Return the tour given by the vertex sequence $(v_1, \dots, v_n)$. Denote by $\mathrm{opt}^{NNH}(K_n, c)$ the shortest possible length of a tour returned by the Nearest Neighbor Heuristic (note that there is some choice involved), and by $\mathrm{opt}(K_n, c)$ the length of an optimum tour. Show that the ratio $\mathrm{opt}^{NNH}(K_n, c)/\mathrm{opt}(K_n, c)$ can be arbitrarily large. (4 points)

Exercise 10.3: As usual, given an instance (K_n, c) of the TSP, denote by $HK(K_n, c)$ the Held-Karp lower bound and by $\mathrm{opt}(K_n, c)$ the length of an optimum tour. Show that there are instances (K_n, c) of the Metric TSP such that the ratio $HK(K_n, c)/\mathrm{opt}(K_n, c)$ is arbitrarily close to $3/4$. (4 points)

Exercise 10.4: **(Updated: A new assumption was added and the points are now bonus points.)** Let $n \geq 3$ and $x : E(K_n) \to [0,1]$. Suppose that for every $v \in V(K_n)$ we have $\sum_{e \in \delta(v)} x_e = 2$ and for every $X \subseteq V(K_n)$ and $F \subseteq \delta(X)$ such that $|F|$ is odd *and F is a matching* we have

$$\sum_{e \in E(K_n[X]) \cup F} x_e \leq |X| + \frac{|F| - 1}{2}$$

Show that x satisfies all of the 2-matching constraints. (4 points)

Deadline: Thursday, January 19, 2017, before the lecture.