Combinatorial Optimization
Exercise set 9

Exercise 9.1: Given a bipartite graph \( G = (A \cup B, E) \), we say that a \( B \)-half-tour in \( G \) is an edge-progression \( v_1, e_1, v_2, \ldots, v_t, e_t, v_{t+1} \) with \( v_1 = v_{t+1} \in A \) and such that every vertex of \( B \) appears precisely once in the sequence. (Vertices of \( A \) may appear any number of times or not at all, and edges may be repeated. And obviously \( t = 2|B| \).)

Given \( k, n \geq 1 \) let \( K_{k,n} \) be the complete bipartite graph with parts of sizes \( k \) and \( n \), i.e., there is a bipartition \( (A_{k,n}, B_{k,n}) \) of \( V(K_{k,n}) \) with \( |A_{k,n}| = k, |B_{k,n}| = n \) and every vertex of \( A_{k,n} \) is connected to every vertex of \( B_{k,n} \). Consider the MINIMUM WEIGHT \( B \)-HALF-TOUR PROBLEM: Given \( k, n \geq 1 \) and an edge-weight function \( c : E(K_{k,n}) \to \mathbb{Q}_{\geq 0} \), find a \( B_{k,n} \)-half-tour in \( K_{k,n} \) of minimum weight. Show that this problem can be solved in time that is polynomial with respect to \( n \) (although exponential with respect to \( k \)).

\( \text{Hint 1:} \) Solve the “simpler” problem in which \( v_1 \) is pre-chosen (given as part of the input).

\( \text{Solving the original problem by solving } k \text{ instances of this “simpler” version is fast enough.} \)

\( \text{Hint 2:} \) Use exercise 8.3 considering the complete graph on \( \{v_1\} \cup B \).

Exercise 9.2: Let \( n \geq 4 \) and \( c : E(K_n) \to \mathbb{R}_{\geq 0} \) be such that \((K_n, c)\) is an instance of the METRIC TSP, and let \( T \) be a tour on \( K_n \). Show that there is a tour \( T' \neq T \) such that

\[ |c(T') - c(T)| \leq \frac{2}{n} \cdot c(T) \]

(4 points)

Exercise 9.3: Let \( n \geq 3 \) and \( x : E(K_n) \to [0,1] \) be such that it satisfies all degree constraints of the TSP but not all subtour elimination constraints. Show that there is a non-empty set \( S \subsetneq V(K_n) \) such that

\[ \sum_{e \in E(K_n[S])} x_e > |S| - 1 \]

and \( x_e < 1 \) for all \( e \in \delta(S) \).

(4 points)

Exercise 9.4: Consider the ANOTHER HAMILTONIAN CIRCUIT PROBLEM: Given an undirected graph \( G \) and a Hamiltonian circuit \( C \) in \( G \), decide whether there is any other Hamiltonian circuit in \( G \). Show that this problem is NP-complete.

(4 points)

Deadline: Thursday, January 12, 2017, before the lecture.

Note: Again, only hard copies this time. No submissions by e-mail!