Winter term 2016/17
Prof. Dr. Stefan Hougardy
Tomás Silveira Salles

Research Institute for Discrete Mathematics
University of Bonn

# Combinatorial Optimization
# Exercise set 8

**Exercise 8.1:** Let $G$ be a simple undirected graph such that $|\delta(v)|$ is odd for every $v \in V(G)$. For an edge $e \in E(G)$, let $\mathcal{H}_e$ denote the family of all Hamiltonian cycles in $G$ containing $e$. Show that $|\mathcal{H}_e|$ is even for every $e \in E(G)$. (4 points)

**Exercise 8.2:** Let $n \geq 3$ and let $f : V(K_n) \to \mathbb{R}^2$ be an injective function. We define a function $g_f : E(K_n) \to \mathcal{P}(\mathbb{R}^2)$ by mapping an edge $\{u, v\}$ to the *open* line-segment between $f(u)$ and $f(v)$ (i.e. the line-segment between $f(u)$ and $f(v)$ without its two endpoints). We say that two edges $e, e' \in E(K_n)$ *cross* (with respect to $f$) if $g_f(e) \cap g_f(e') \neq \emptyset$.

Given a Hamiltonian tour $T$ in $K_n$ and two edges $e, e' \in E(T)$ which cross, consider the algorithm $\textsc{RemoveCrossing}(T, e, e')$: Let $u, v$ be the endpoints of $e$ and $u', v'$ be the endpoints of $e'$. Delete $e$ and $e'$ from $T$. If $T + \{u, u'\} + \{v, v'\}$ is a Hamiltonian tour, add $\{u, u'\}$ and $\{v, v'\}$ to $T$. Otherwise add $\{u, v'\}$ and $\{v, u'\}$ to $T$.

(i) Show that after $\textsc{RemoveCrossing}(T, e, e')$, $T$ is still a Hamiltonian tour. (1 point)

Consider the algorithm $\textsc{RemoveAllCrossings}(T)$: While there are edges in $T$ that cross, choose one such pair of edges $e, e' \in E(T)$ and call $\textsc{RemoveCrossing}(T, e, e')$.

(ii) Assuming that in the range of $f$ no three points are colinear (i.e. lie on the same straight-line), show that for any Hamiltonian tour $T$ the algorithm $\textsc{RemoveAllCrossings}(T)$ terminates after $O(n^3)$ calls of $\textsc{RemoveCrossing}$. (4 points)

(iii) Give an example (i.e. a choice of $n$, $f$ and $T$) such that $\textsc{RemoveAllCrossings}(T)$ does not terminate. (1 point)

**Exercise 8.3:** Let $n \geq 3$ and consider the complete graph $K_n$ with an edge-weight function $c : E(K_n) \to \mathbb{Q}_{\geq 0}$. Let $M_1, \ldots, M_k \subseteq V(K_n)$ be disjoint sets such that $M_1 \dot{\cup} \ldots \dot{\cup} M_k = V(K_n)$ and

$$\forall i\ \forall u, v \in M_i \text{ distinct: } \ c(\{u, v\}) = 0$$

(i) Show that every optimal solution of the TSP for $(K_n, c)$ uses at most $k(k-1)$ edges of strictly positive weight. (2 points)

(ii) Show that there is an edge-weight function $c' : E(K_n) \to \mathbb{Q}_{\geq 0}$ such that

$$\forall i\ \forall u, v \in M_i \text{ distinct: } \ c'(\{u, v\}) = 0$$

and

$$\forall i, j \text{ distinct } \forall u \in M_i\, \forall v \in M_j : \ c'(\{u, v\}) > 0$$

and such that every optimum solution of the TSP for $(K_n, c')$ is also an optimum solution of the TSP for $(K_n, c)$. (2 points)

(iii) Show that (using the family $M_1, \ldots, M_k$) we can find an optimal solution of the TSP for $(K_n, c)$ in $O(n^{2k(k-1)+1})$-time. (3 points)

**Deadline:** Thursday, December 22, 2016, before the lecture.

**Note: Only hard copies this time. No submissions by e-mail!**