Winter term 2016/17
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Combinatorial Optimization
Exercise set 7

Exercise 7.1: Let G be a graph and $T \subseteq V(G)$ with $|T|$ even. Prove:

(i) A set $F \subseteq E(G)$ intersects every T-join if and only if it contains a T-cut. (2 points)

(ii) A set $F \subseteq E(G)$ intersects every T-cut if and only if it contains a T-join. (2 points)

Exercise 7.2: Let G be a bipartite graph and $J \subseteq E(G)$. Consider the properties:

(i) $|J \cap E(C)| \leq (1/2) \cdot |E(C)|$ for every circuit C in G.

(ii) There exist $|J|$ pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_{|J|})$ with $|\delta(X_i) \cap J| = 1$ for every i.

Show that (i) and (ii) are equivalent. (4 points)

Exercise 7.3: The Undirected Minimum Mean-Weight Cycle Problem is the following: Given an undirected graph G with edge-weights $c : E(G) \to \mathbb{R}$, find a cycle C whose mean-weight $c(E(C))/|E(C)|$ is minimum, or determine that G is acyclic. Consider the following algorithm for the Undirected Minimum Mean-Weight Cycle Problem: First determine with a linear search whether G has cycles or not, and if not return with this information. Let $\gamma := \max\{c(e) : e \in E(G)\}$ and define a new edge-weight function via $c'(e) := c(e) - \gamma$. Let $T := \emptyset$. Now iterate the following: Find a minimum c'-weight T-join J with a polynomial (black-box) algorithm. If $c'(J) = 0$, return any zero-c'-weight cycle. Otherwise, let $\gamma' := c'(J)/|J|$, reset c' via $c'(e) \leftarrow c'(e) - \gamma'$, and continue.

Show that this algorithm works correctly and runs in polynomial time. Also, explain how to the get the cycle to be returned in the case $c'(J) = 0$. (4 points)

Exercise 7.4: For an undirected graph G, let P_G denote the spanning-tree polytope of G and

$$Q_G := \left\{ x : E(G) \to [0,1] \; : \; \sum_{e \in E(G)} x(e) = |V(G)| - 1 \text{ and } \sum_{e \in \delta(X)} x(e) \geq 1 \text{ for } \emptyset \neq X \subsetneq V(G) \right\}$$

(i) Show that $P_G \subseteq Q_G$ for every G. (1 point)

(ii) Show that $P_G = Q_G$ if G is acyclic. (1 point)

(iii) Show that $P_G = Q_G$ if G is a cycle. (2 points)

(iv) Give an example of a graph G such that $P_G \neq Q_G$. (2 points)

Deadline: Thursday, December 15, 2016, before the lecture.