Winter term 2016/17
Prof. Dr. Stefan Hougardy
Tomás Silveira Salles

Research Institute for Discrete Mathematics
University of Bonn

Combinatorial Optimization
Exercise set 5

Exercise 5.1: Recall exercise 2.3. Although you only had to solve parts (vi) and (vii) for bipartite graphs, they can also be solved in general for undirected graphs (with the same runtime), only with a more complicated algorithm in part (vi). Using this, describe a linear-time approximation scheme for the MAXIMUM CARDINALITY MATCHING PROBLEM. More specifically, describe an algorithm which takes as input an undirected graph G and a positive number ε, outputs a matching M in G such that $|M| \geq (1-\varepsilon)\cdot\nu(G)$, and runs in

$$O(\frac{1}{\varepsilon}\cdot(|V(G)|+|E(G)|))\text{-time.}$$

(4 points)

Exercise 5.2: Let G be an undirected graph with edge weights $c : E(G) \to \mathbb{R}$. A *fractional vertex-cover* in (G,c) is a function $w : V(G) \to \mathbb{R}_{\geq 0}$ satisfying

$$\forall\, e = \{u,v\} \in E(G) \quad w(u) + w(v) \geq c(e)$$

The *size* of w is defined as $\texttt{size}(w) := \sum_{v\in V(G)} w(v)$. Let $\nu(G,c)$ denote the maximum weight of a matching in G, and let $\tau(G,c)$ denote the minimum size of a fractional vertex-cover in (G,c). Show that:

(i) $\nu(G,c) \leq \tau(G,c) \leq 2\cdot\nu(G,c)$. (2 points)

(ii) If $c \equiv 1$, then $\nu(G,c) = \nu(G)$ and $\tau(G,c) \leq \tau(G)$. Give an example to show that even when $c \equiv 1$ we can still have $\tau(G,c) < \tau(G)$. (2 points)

(iii) $\nu(G,c) = \tau(G,c)$ if G is bipartite. (2 points)

Exercise 5.3: Let $k \in \mathbb{N}$, $k \geq 1$, and suppose G is a k-regular and $(k-1)$-edge-connected graph with an even number of vertices, and with edge weights $c : E(G) \to \mathbb{R}$. Show that there is a perfect matching M in G with $c(M) \leq (1/k)\cdot c(E(G))$. (4 points)

Exercise 5.4: Let G be a bipartite graph with edge weights $c : E(G) \to \mathbb{R}$, let $k \in \mathbb{N}$ and let M be a matching in G with $|M| = k$ that has minimum weight among all matchings in G that contain exactly k edges. Let P be an M-augmenting path with minimum gain among all M-augmenting paths (where $\texttt{gain}(P,M) := c(E(P)\setminus M) - c(E(P)\cap M)$) and finally let $M' := M\Delta E(P)$. Show that M' has minimum weight among all matchings in G that contain exactly $k+1$ edges. (4 points)

Deadline: Thursday, November 24, 2016, before the lecture.