Winter term 2016/17
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# Combinatorial Optimization
## Exercise set 4

**Exercise 4.1:** (Infinite counter-example to Hall's Theorem) Find a countably infinite bipartite graph $G = (A\dot{\cup}B, E)$ with $|N(X)| \geq |X|$ for every $X \subseteq A$, such that there is no matching covering $A$ in $G$. (2 points)

**Exercise 4.2:** Let $S = \{1, \ldots, n\}$ for some $n \geq 1$.

(i) Suppose $0 \leq k \leq n-1$ and consider the bipartite graph $G = (A\dot{\cup}B, E)$ where

$$\begin{aligned} A &:= \{X \subseteq S \,:\, |X| = k\} \\ B &:= \{Y \subseteq S \,:\, |Y| = k+1\} \\ E &:= \{\{X, Y\} \,:\, X \in A,\, Y \in B,\, X \subseteq Y\} \end{aligned}$$

Show that there is a matching covering $A$ if $k < n/2$, and that there is a matching covering $B$ if $k > n/2 - 1$. (1 point)

(ii) Suppose $\mathcal{F}$ is a family of subsets of $S$ with the property that no element of $\mathcal{F}$ is contained in another element of $\mathcal{F}$. Show that:

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

and that this bound is tight (for every $n$). (3 points)

**Exercise 4.3:** Consider the Bottleneck Matching Problem: Given an undirected graph $G$ with edge weights $c : E(G) \to \mathbb{R}$, find a perfect matching $M$ (if one exists) minimizing $\max\{c(e) \,:\, e \in M\}$. Show how to solve the Bottleneck Matching Problem in $O(nm \log n)$-time. (4 points)

**Exercise 4.4: (Cancelled!)** Let $G$ be a bipartite graph with edge weights $c : E(G) \to \mathbb{R}$, let $k \in \mathbb{N}$ and let $M$ be a matching in $G$ with $|M| = k$ that has minimum weight among all matchings in $G$ that contain exactly $k$ edges. Let $P$ be an $M$-augmenting path with minimum <weight> (wrong, should be "gain") among all $M$-augmenting paths and finally let $M' := M \Delta E(P)$. Show that $M'$ has minimum weight among all matchings in $G$ that contain exactly $k+1$ edges. (4 points)

**Exercise 4.5:** Let $G$ be a graph with edge weights $c : E(G) \to \mathbb{R}_{\geq 0}$. Let $\nu(G, c)$ denote the maximum weight of a matching in $G$. Suppose $M$ is a matching in $G$ for which there is no 2-augmentation with strictly positive gain. Show that

$$c(M) \geq \frac{2}{3}\,\nu(G, c)$$

(2 points)

**Note on exercise 4.5:** In the printed version you got I mistakenly wrote "bipartite graph". The assumption that $G$ is bipartite is unnecessary, and probably does not make the solution any easier.

**Deadline:** Thursday, November 17, 2016, before the lecture.