Winter term 2016/17
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Combinatorial Optimization
Exercise set 2

Note: Remember that exercise 1.3 (from the last set) is also due on November 3!

Exercise 2.1: Let G be a bipartite graph with no isolated vertices. Show that the minimum cardinality of an edge-cover of G is equal to the maximum cardinality of a stable set in G.

(4 points)

Exercise 2.2: Let G be a graph with n vertices where each vertex has at least $n/2$ many neighbors. Show that $\nu(G) = \lfloor n/2 \rfloor$. (4 points)

Exercise 2.3: Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Prove that there exists an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M\Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)}+1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\begin{aligned}&\texttt{min}\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\}\\ &\qquad > \texttt{min}\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}\end{aligned}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(8 points)

Deadline: Thursday, November 3, 2016, before the lecture.