Combinatorial Optimization
Exercise set 2

Note: Remember that exercise 1.3 (from the last set) is also due on November 3!

Exercise 2.1: Let $G$ be a bipartite graph with no isolated vertices. Show that the minimum cardinality of an edge-cover of $G$ is equal to the maximum cardinality of a stable set in $G$. (4 points)

Exercise 2.2: Let $G$ be a graph with $n$ vertices where each vertex has at least $n/2$ many neighbors. Show that $\nu(G) = \lfloor n/2 \rfloor$. (4 points)

Exercise 2.3: Let $G$ be a graph and $M$ a matching in $G$ that is not maximum. In this exercise we use the terminology disjoint subgraphs/paths/circuits and mean it quite literally: Two subgraphs are disjoint if they have no edges and no vertices in common. (Note that the term vertex-disjoint paths is often used to mean that two paths have no inner-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint $M$-augmenting paths in $G$.

(ii) Prove that there exists an $M$-augmenting path of length at most $\nu(G) + |M| / \nu(G) - |M|$. (4 points)

(iii) Let $P$ be a shortest $M$-augmenting path in $G$ and $P'$ an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$. (4 points)

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then $P_i$ and $P_j$ are disjoint. (4 points)

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers. (4 points)

From now on, let $G$ be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching $M$ in $G$ show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest $M$-augmenting paths such that if $M'$ is the matching obtained by augmenting $M$ over every path in $\mathcal{P}$, then

$$\min\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\} > \min\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n(m+n)})$ that solves the CARDINALITY MATCHING PROBLEM in bipartite graphs. (8 points)

Deadline: Thursday, November 3, 2016, before the lecture.