Winter term 2015/16 Research Institute for Discrete Mathematics
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Linear and Integer Optimization
Assignment Sheet 9

1. Let G be simple undirected graph. Consider the following linear program:

$$\begin{array}{llcll}
\min & \sum_{e=\{v,w\}\in E(G)} x_{vw} & & & \\
\text{s.t.} & \sum_{w\in S} x_{vw} & \geq & \lceil \frac{1}{4}|S|^2 + \frac{1}{2}|S| \rceil & \text{for } v \in V(G), S \subseteq V(G)\setminus\{v\} \\
 & x_{uw} & \leq & x_{uv} + x_{vw} & \text{for } u, v, w \in V(G) \\
 & x_{vw} & \geq & 0 & \text{for } v \in V(G) \\
 & x_{vv} & = & 0 & \text{for } v \in V(G)
\end{array}$$

   (a) Show that this is an LP-relaxation of the following problem: Find distances x_{vw} for the nodes of G such that $\sum_{e=\{v,w\}\in E(G)} x_{vw}$ is minimized under the condition that there is an ordering $\{v_1, \ldots, v_{|V(G)|}\} = V(G)$ with $x_{v_i v_j} = |i-j|$ for $i, j \in \{1, \ldots, |V(G)|\}$.

   (b) Prove that there is a polynomial-time separation oracle for the polyhedron of the feasible solutions of the LP. (4+4 points)

2. Consider the following optimization problem:

$$\begin{array}{ll}
\min & \frac{c^t x + d}{f^t x + g} \\
\text{s.t.} & Ax \leq b \\
 & \|x\|_\infty \leq R
\end{array}$$

   where $c, f \in \mathbb{Q}^n$, $d, g, R \in \mathbb{Q}$, $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$. You may assume that $f^t x + g > 0$ and $c^t x + d > 0$ for any $x \in \mathbb{R}^n$ with $\|x\|_\infty \leq R$ and that there is a feasible solution. Show that for any $\epsilon > 0$ there is a polynomial-time algorithm computing a feasible solution x^* with $\frac{c^t x^* + d}{f^t x^* + g} \leq \mathrm{OPT}(1+\epsilon)$ where OPT is the value of an optimum solution. (6 points)

3. Consider the following situation: You are a foreign exchange trader trading in n different currencies with an initial budget $b = (b_1, \ldots, b_n) \geq 0$, where b_i is the amount of currency i. For every pair $1 \leq i, j \leq n$ of currencies, you are given an exchange rate r_{ij}. Assume that you are able to exchange an arbitrary (non-negative) amount x_{ij} of currency i into $r_{ij} x_{ij}$ units of currency j. You want to determine trades that will maximize the amount of currency 1 available to you in the end. You may assume that all trades can occur simultaneously (e.g. by cost-free borrowing for the instant during which the trades occur), so long as the final amounts in all currencies are non-negative.

   Model this problem as a linear program and derive its dual. (5 points)

   Remark: If the LP is unbounded, one speaks of an *arbitrage opportunity*, i.e. a situation that allows "risk-free" profit.

4. A parcel service leases cars on a basis of 3, 4 or 5 months. The cost for a 3-months leasing contract (for one car) is 1700 EUR, for a 4-months contract 2200 EUR, and for 5-months contract 2500 EUR. For a certain period (e.g. one year), the company knows in advance for each month how many vehicles will be needed in that month. Formulate the problem of finding a cheapest way to lease sufficiently many cars as a linear program. Show in particular that the linear program always has an optimum solution that is integral. (6 points)

Due date: Tuesday, January 12, 2016, before the lecture.