Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization
Assignment Sheet 6

1. Describe an algorithm for the following problem: Given a tree T, you have time $O(|V(T)|)$ for some preprocessing. After the preprocessing, you should be able to compute for any two given nodes x and y of T in time $O(\mathrm{dist}_T(x,y))$ the x-y-path in T. (4 points)
Remark: This is a problem that has to be solved during the NETWORK SIMPLEX ALGORITHM when computing a fundamental circuit.

2. Let (G, u, b, c) be an instance of the MINIMUM-COST FLOW PROBLEM.

   (a) Dualize the linear program formulation of the MINIMUM-COST FLOW PROBLEM that was presented in the lecture.

   (b) Let (r, T, L, U) be a feasible spanning tree structure for (G, u, b, c), and let f be the flow and π the potential associated to it. Show by considering the complementary slackness constraints that f is optimum if $c_\pi(e) \geq 0$ for all $e \in L$ and $c_\pi(e) \leq 0$ for all $e \in U$. (3+3 points)

   Remark: The statement in (b) has already been proved in a different way in the lecture (see Proposition 37 of the lecture notes).

3. Consider the following linear program:

$$
\begin{array}{lrcll}
\min & \sum_{i=1}^{n} c_i x_i & & & \\
\text{s.t.} & \sum_{i=1}^{n} a_i x_i & = & b & \\
& x_i & \leq & 1 & \text{for } i = 1, \ldots, n \\
& x_i & \geq & 0 & \text{for } i = 1, \ldots, n
\end{array}
$$

   (a) Describe a simple and efficient feasibility test for the problem.

   (b) Give an algorithm with running time $O(n \log n)$ that finds an optimum solution. (5 points)

4. Proof that for $r_1, \ldots, r_n \in \mathbb{Q}$, we have

   (a) $\operatorname{size}\left(\prod_{i=1}^{n} r_i\right) \leq \sum_{i=1}^{n} \operatorname{size}(r_i)$

   (b) $\operatorname{size}\left(\sum_{i=1}^{n} r_i\right) \leq 2 \sum_{i=1}^{n} \operatorname{size}(r_i)$ (1+1 points)

Due date: Tuesday, December 8, 2015, before the lecture.