Linear and Integer Optimization
Assignment Sheet 6

1. Describe an algorithm for the following problem: Given a tree $T$, you have time $O(|V(T)|)$ for some preprocessing. After the preprocessing, you should be able to compute for any two given nodes $x$ and $y$ of $T$ in time $O(\text{dist}_T(x,y))$ the $x$-$y$-path in $T$. (4 points)

**Remark:** This is a problem that has to be solved during the network simplex algorithm when computing a fundamental circuit.

2. Let $(G,u,b,c)$ be an instance of the MINIMUM-COST FLOW PROBLEM.

   (a) Dualize the linear program formulation of the MINIMUM-COST FLOW PROBLEM that was presented in the lecture.

   (b) Let $(r,T,L,U)$ be a feasible spanning tree structure for $(G,u,b,c)$, and let $f$ be the flow and $\pi$ the potential associated to it. Show by considering the complementary slackness constraints that $f$ is optimum if $c_\pi(e) \geq 0$ for all $e \in L$ and $c_\pi(e) \leq 0$ for all $e \in U$. (3+3 points)

**Remark:** The statement in (b) has already been proved in a different way in the lecture (see Proposition 37 of the lecture notes).

3. Consider the following linear program:

   $$\min \sum_{i=1}^{n} c_i x_i$$

   s.t. $\sum_{i=1}^{n} a_i x_i = b$

   $$x_i \leq 1 \quad \text{for } i = 1, \ldots, n$$

   $$x_i \geq 0 \quad \text{for } i = 1, \ldots, n$$

   (a) Describe a simple and efficient feasibility test for the problem.

   (b) Give an algorithm with running time $O(n \log n)$ that finds an optimum solution. (5 points)

4. Proof that for $r_1, \ldots, r_n \in \mathbb{Q}$, we have

   (a) $\text{size} \left( \prod_{i=1}^{n} r_i \right) \leq \sum_{i=1}^{n} \text{size}(r_i)$

   (b) $\text{size} \left( \sum_{i=1}^{n} r_i \right) \leq 2 \sum_{i=1}^{n} \text{size}(r_i)$ (1+1 points)

Due date: Tuesday, December 8, 2015, before the lecture.