Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 12

Exercise 12.1:
Let $f : 2^U \to \mathbb{R}$ be a submodular function. Let R be a random subset of U, where each element is chosen independently with probability $\frac{1}{2}$. Prove:

(a) $\mathbb{E}(f(R)) \geq \frac{1}{2}(f(\emptyset) + f(U))$.

(b) For each $A \subseteq U$ we have $\mathbb{E}(f(R)) \geq \frac{1}{4}(f(\emptyset) + f(A) + f(U \setminus A) + f(U))$.

(c) If f is nonnegative, then $\mathbb{E}(f(R)) \geq \frac{1}{4} \max_{A \subseteq U} f(A)$.

Note: Part (c) implies a randomized 4-factor approximation algorithm for (nonnegative) submodular function maximization. This problem cannot be solved optimally with a polynomial number of oracle calls. (4 Points)

Exercise 12.2:
Show that if we place the randomized step in the SIMPLE SUBMODULAR FUNCTION MAXIMIZATION ALGORITHM by setting $A := A \cup \{i\}$ if $\Delta_A \geq \Delta_B$ and $B := B \setminus \{i\}$ otherwise yields a 3-approximation (deterministically, in linear time). (4 Points)

Exercise 12.3:
Consider the problem of finding an extreme point in the polytope defined in the proof of Buchbinder and Feldmann's deterministic polynomial-time 2-approximation algorithm for the problem of maximizing a nonnegative submodular function.

(a) Replace one of the two nontrivial constraints by an objective function. Use this to show that there is an extreme point in which at most one variable has a fractional value.

(b) Show how to solve the LP resulting from (a) in $O(n \log n)$ time (note that $|T_{i-1}| = O(n)$).

(4 Points)

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Exercise 12.4:
For any METRIC TSP instance (K_n, c), show that the integrality gap of the subtour-LP is at most $\frac{3}{2}$. The subtour-LP is given by

$$\begin{aligned}
\min \sum_{e \in E} c_e x_e & & \\
\text{s.t. } x_e \in [0,1] & & (e \in E(K_n)), \\
\sum_{e \in \delta(v)} x_e = 2 & & (v \in V(K_n)), \\
\sum_{e \in E(K_n[X])} x_e \leq |X| - 1 & & (\emptyset \neq X \subset V(K_n)),
\end{aligned}$$

and the integrality gap of a linear program is defined as the ratio of the cost of an optimum integral solution to the cost of an optimum (potentially fractional) solution.
Hint: Recall Christofides' Algorithm. Show that for any vector x in the subtour polytope, $\frac{1}{2}x$ is in the T-join polyhedron for every even set T. (4 Points)

Deadline: Tuesday, February 2, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.