Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 11

Exercise 11.1:
Let $f : 2^E \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Prove:

$$\begin{aligned}
&\min \{f(X) : X \subseteq E\} \\
=& \max \left\{ \sum_{u\in E} z(u) : z \leq 0, z(A) \leq f(A) \quad (A \subseteq E) \right\} \\
=& \max \left\{ \sum_{u\in E} \min\{0, y(u)\} : y(A) \leq f(A) \quad (A \subseteq E)\,, y(E) = f(E) \right\}\,.
\end{aligned}$$

(3 Points)

Exercise 11.2:
Let $f : E \to \mathbb{R}$ with $f(\emptyset) = 0$ be a submodular function and x^* a minimizer of

$$\min \sum_{e\in E} x(e)^2$$
$$\text{subject to } x \in B(f)\,,$$

where $B(f)$ is the base polyhedron of f. Let

$$\begin{aligned}
y^* : E \to \mathbb{R}, e &\mapsto \min\{x^*(e), 0\} \\
A_- &:= \{e \in E \mid x^*(e) < 0\}, \\
A_0 &:= \{e \in E \mid x^*(e) \leq 0\}\,.
\end{aligned}$$

Show that

(i) y^* maximizes $\{\sum_{u\in E} y(u) : y \leq 0, y(A) \leq f(A) \quad (A \subseteq E)\}$,

(ii) A_- is the unique minimizer of f and A_0 is the unique maximal minimizer of f.

(3 Points)

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Exercise 11.3:
Prove that the set of vertices of the base polyhedron of a submodular function $f : 2^E \to \mathbb{R}$ with $f(\emptyset) = 0$ is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of E, where

$$b^{\prec}(e) := f(\{v \in E : v \preceq e\}) - f(\{v \in E : v \prec e\}) \quad (e \in E).$$

(3 Points)

Exercise 11.4:
Let $G = (V, E)$ be an undirected graph. For a set $X \subseteq V$ let $f(X)$ denote the number of edges in E with at least one end point in X.

(i) Prove that f is a submodular function.

Let $y : V \to \mathbb{N}$ be a function. We want to find an orientation of G (i.e. a directed graph $G' = (V', E')$ such that $V' = V$ and the underlying undirected graph is equal to G) such that $|\delta^-(v)| = y(v)$ for each $v \in V$.

(ii) Show that such an orientation exists if and only if

$$y(V) = |E| \quad \text{and} \quad y(X) \leq f(X)\ \forall X \subseteq V.$$

Hint: Construct a network (V', E', u) with

$$\begin{aligned} V' &:= E \cup V \cup \{s, t\}, \\ E' &:= \{(s, e) \mid e \in E\} \cup \{(e, v) \mid e \in E, v \in e\} \cup \{(v, t) \mid v \in V\} \end{aligned}$$

and a suitable capacity function u.

(iii) Give a polynomial time combinatorial algorithm which either finds an orientation as desired or a set $X \subset V$ which serves as certificate that such an orientation does not exist.

(iv) Consider the following alternative algorithm: The question if an orientation with the required property exists can be answered by using Schrijver's algorithm to find a set X minimizing the submodular function $f(X) - y(X)$. If this minimum is negative return the minimizer X as there is no orientation as desired. Otherwise start with $G' = (V, \emptyset)$. For each edge $e = \{v, w\} \in E(G)$, set $G' := G' + (v, w)$, $G := G - e$, and $y(w) := y(w) - 1$ if there is an orientation satisfying $|\delta^-(v)| = y(v)$ for each $v \in V$ for $G - e$ after decreasing $y(w)$ by 1, otherwise set $G' := G' + (w, v)$, $G := G - e$, and $y(v) := y(v) - 1$.

Proof the correctness of this algorithm and compare its runtime to the runtime of your algorithm from part (iii).

(1+3+1+2 Points)

Deadline: Tuesday, January 26, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.