Combinatorial Optimization

Exercise Sheet 11

Exercise 11.1:
Let $f : 2^E \rightarrow \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Prove:

$$\min \{ f(X) : X \subseteq E \} = \max \left\{ \sum_{u \in E} z(u) : z \leq 0, z(A) \leq f(A) \quad (A \subseteq E) \right\}$$

$$= \max \left\{ \sum_{u \in E} \min \{0, y(u)\} : y(A) \leq f(A) \quad (A \subseteq E), y(E) = f(E) \right\}.$$ (3 Points)

Exercise 11.2:
Let $f : E \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$ be a submodular function and $x^*$ a minimizer of

$$\min \sum_{e \in E} x(e)^2$$

subject to $x \in B(f)$,

where $B(f)$ is the base polyhedron of $f$. Let

$$y^* : E \rightarrow \mathbb{R}, e \mapsto \min \{x^*(e), 0\}$$

$$A_- := \{ e \in E \mid x^*(e) < 0 \},$$

$$A_0 := \{ e \in E \mid x^*(e) \leq 0 \}.$$

Show that

(i) $y^*$ maximizes $\{ \sum_{u \in E} y(u) : y \leq 0, y(A) \leq f(A) \quad (A \subseteq E) \}$,

(ii) $A_-$ is the unique minimizer of $f$ and $A_0$ is the unique maximal minimizer of $f$. (3 Points)

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Exercise 11.3:
Prove that the set of vertices of the base polyhedron of a submodular function \( f : 2^E \to \mathbb{R} \) with \( f(\emptyset) = 0 \) is precisely the set of vectors \( b^\prec \) for all total orders \( \prec \) of \( E \), where
\[
b^\prec(e) := f(\{v \in E : v \preceq e\}) - f(\{v \in E : v \prec e\}) \quad (e \in E).
\]

(3 Points)

Exercise 11.4:
Let \( G = (V,E) \) be an undirected graph. For a set \( X \subseteq V \) let \( f(X) \) denote the number of edges in \( E \) with at least one end point in \( X \).

(i) Prove that \( f \) is a submodular function.

Let \( y : V \to \mathbb{N} \) be a function. We want to find an orientation of \( G \) (i.e. a directed graph \( G' = (V',E') \) such that \( V' = V \) and the underlying undirected graph is equal to \( G \)) such that \( |\delta^-(v)| = y(v) \) for each \( v \in V \).

(ii) Show that such an orientation exists if and only if
\[
y(V) = |E| \quad \text{and} \quad y(X) \leq f(X) \quad \forall X \subseteq V.
\]

Hint: Construct a network \((V',E',u)\) with
\[
V' := E \cup V \cup \{s,t\},
E' := \{(s,e) \mid e \in E\} \cup \{(e,v) \mid e \in E, v \in e\} \cup \{(v,t) \mid v \in V\}
\]
and a suitable capacity function \( u \).

(iii) Give a polynomial time combinatorial algorithm which either finds an orientation as desired or a set \( X \subset V \) which serves as certificate that such an orientation does not exist.

(iv) Consider the following alternative algorithm: The question if an orientation with the required property exists can be answered by using Schrijver’s algorithm to find a set \( X \) minimizing the submodular function \( f(X) - y(X) \). If this minimum is negative return the minimizer \( X \) as there is no orientation as desired. Otherwise start with \( G' = (V,\emptyset) \). For each edge \( e = \{v, w\} \in E(G) \), set \( G' := G' + (v, w) \), \( G := G - e \), and \( y(w) := y(w) - 1 \) if there is an orientation satisfying \( |\delta^-(v)| = y(v) \) for each \( v \in V \) for \( G - e \) after decreasing \( y(w) \) by 1, otherwise set \( G' := G' + (w, v) \), \( G := G - e \), and \( y(v) := y(v) - 1 \).

Proof the correctness of this algorithm and compare its runtime to the runtime of your algorithm from part (iii).

(1+3+1+2 Points)

Deadline: Tuesday, January 26, 2015, before the lecture.
Information: Submissions by groups of up to three students are allowed.