Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 10

Exercise 10.1:
Let $E = \{1, 2, \ldots, n\}$. Then the *permutahedron* P_n is defined as

$$P_n := \text{conv}\left(\{\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{R}^n \mid \sigma \text{ permutation of } E\}\right) .$$

Find a submodular function $\rho : 2^E \to \mathbb{Z}$ with $\rho(\emptyset) = 0$, s.t. the base polyhedron $B(\rho)$ is the permutahedron P_n. (4 Points)

Exercise 10.2:
Let G be a digraph, $s, t \in V(G), u : E(G) \to \mathbb{R}_+$, and $A := \delta^+(s)$. Let

$$P := \{x \in \mathbb{R}^A : \text{ there is an } s\text{-}t\text{-flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in A\} .$$

Prove that P is a polymatroid. (4 Points)

Exercise 10.3:
Let S be a finite set and $f : 2^S \to \mathbb{R}$. Define $f' : \mathbb{R}^S_+ \to \mathbb{R}$ as follows. For any $x \in \mathbb{R}^S_+$, there are unique $k \in \mathbb{Z}_+, \lambda_1, \ldots, \lambda_k > 0$ and $\emptyset \subset T_1 \subset T_2 \subset \ldots \subset T_k \subseteq S$ such that $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$, where χ^{T_i} is the incidence vector of T_i. Then $f'(x) := \sum_{i=1}^k \lambda_i f(T_i)$.
Prove that f is submodular if and only if f' is convex. (4 Points)

Exercise 10.4:
Prove that a nonempty compact set $P \subseteq \mathbb{R}^n_+$ is a polymatroid if and only if

1. For all $0 \leq x \leq y \in P$ we have $x \in P$.
2. For all $x \in \mathbb{R}^n_+$ and all $y, z \leq x$ with $y, z \in P$ that are maximal with this property (i.e. $y \leq w \leq x$ and $w \in P$ implies $w = y$, and $z \leq w \leq x$ and $w \in P$ implies $w = z$) we have $\mathbb{1}y = \mathbb{1}z$.

Note: This is the original definition of Edmonds. (4 Points)

Deadline: Tuesday, January 19, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.