Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 9

Exercise 9.1: Prove Tutte's perfect b-matching characterization. Let G be an undirected graph, $u : E(G) \to \mathbb{N} \cap \{\infty\}$, and $b : V(G) \to \mathbb{N}$. (G, u) has a perfect b-matching if and only if for any two subsets $X, Y \subset V(G)$ with $X \cap Y = \emptyset$, the number of connected components C in $G - X - Y$ for which $\sum_{v \in V(C)} b(c) + \sum_{e \in E(V(C),Y)} u(e)$ is odd is upper bounded by

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E(X,Y)} u(e)\,.$$

(5 Points)

Exercise 9.2: Let G be a graph, $b : V(G) \to \mathbb{N}$, and $c : E(G) \to \mathbb{R}$ a weight function.

1. Show that the uncapacitated maximum-weight b-matching problem in bipartite graphs can be solved in strongly polynomial time.
2. Use Step 1 to show that the uncapacitated maximum-weight b-matching problem can be solved in strongly polynomial time if b is even.
3. Use Step 2 to show that the uncapacitated maximum-weight b-matching problem can be solved in strongly polynomial time.
4. Use Step 3 to show that the capacitated maximum-weight b-matching problem for edge capacities $u : E(G) \to \mathbb{N} \cup \{\infty\}$ can be solved in strongly polynomial time.

(2+1+1+1 Points)

Exercise 9.3: Let U be a finite set and $f : 2^U \to \mathbb{R}$. Prove that f is submodular if and only if $f(X \cup \{y, z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq U$ and $y, z \in U$ with $y \neq z$.

(3 Points)

Continued on next page.

Exercise 9.4: Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$. Let $f : 2^A \to \mathbb{N}$ be defined by

$$f(X) = \nu(G[X \cup \Gamma(X)]), \quad X \subseteq A\,.$$

Show that f is submodular.

(3 Points)

Deadline: Tuesday, January 12, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.