Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 5

Exercise 5.1: Let $G = (V, E)$ be an undirected graph and Q its fractional perfect matching polytope, which is defined by

$$Q = \{x \in \mathbb{R}^E : x_e \geq 0 \ (e \in E), \sum_{e \in \delta(v)} x_e = 1 \ (v \in V)\}.$$

Prove that a vector $x \in Q$ is a vertex of Q if and only if there exist vertex disjoint odd circuits $C_1, \ldots, C_k$ and a perfect matching M in $G - (V(C_1) \cup \ldots \cup V(C_k))$ such that

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k), \\ 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

These vertices are called half-integral. (4 Points)

Exercise 5.2: Let G be a k-regular and $(k-1)$-edge-connected graph with an even number of vertices, and let $c : E(G) \to \mathbb{R}_+$. Prove that there exists a perfect matching M in G with $c(M) \geq \frac{1}{k} c(E(G))$.
Hint: Use the perfect matching polytope.

Exercise 5.3: Let c_{ij} be costs on the edges of the complete graph K_{2n+1}. A graph with $2n+1$ vertices is called a *double star* if it emerges from a star with $n+1$ vertices by replacing every edge $\{v, w\}$ by a vertex z_{vw} and two edges $\{v, z_{vw}\}, \{z_{vw}, w\}$. Show that there exists a polynomial time algorithm to find a spanning double star of K_{2n+1} with minimum cost. (4 Points)

Exercise 5.4: Let $G = (V, E)$ be an undirected graph and $n := |V|$. Prove that the spanning tree polytope of G is in general a proper subset of the polytope

$$\{x \in [0,1]^E : \sum_{e \in E} x_e = n - 1, \sum_{e \in \delta(X)} x_e \geq 1 \text{ for } \emptyset \neq X \subset V\}.$$

(4 Points)

Deadline: Tuesday, December 1, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.