Exercise 5.1: Let $G = (V, E)$ be an undirected graph and $Q$ its fractional perfect matching polytope, which is defined by

$$Q = \{ x \in \mathbb{R}^E : x_e \geq 0 (e \in E), \sum_{e \in \delta(v)} x_e = 1 (v \in V) \}.$$ 

Prove that a vector $x \in Q$ is a vertex of $Q$ if and only if there exist vertex disjoint odd circuits $C_1, \ldots, C_k$ and a perfect matching $M$ in $G - (V(C_1) \cup \ldots \cup V(C_k))$ such that

$$x_e = \begin{cases} 
\frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k), \\
1 & \text{if } e \in M, \\
0 & \text{otherwise.}
\end{cases}$$

These vertices are called half-integral. (4 Points)

Exercise 5.2: Let $G$ be a $k$-regular and $(k - 1)$-edge-connected graph with an even number of vertices, and let $c : E(G) \to \mathbb{R}_+$. Prove that there exists a perfect matching $M$ in $G$ with $c(M) \geq \frac{1}{k}e(G)$.

*Hint:* Use the perfect matching polytope.

(4 Points)

Exercise 5.3: Let $c_{ij}$ be costs on the edges of the complete graph $K_{2n+1}$. A graph with $2n+1$ vertices is called a double star if it emerges from a star with $n+1$ vertices by replacing every edge $\{v, w\}$ by a vertex $z_{vw}$ and two edges $\{v, z_{vw}\}, \{z_{vw}, w\}$. Show that there exists a polynomial time algorithm to find a spanning double star of $K_{2n+1}$ with minimum cost. (4 Points)
Exercise 5.4: Let $G = (V, E)$ be an undirected graph and $n := |V|$. Prove that the spanning tree polytope of $G$ is in general a proper subset of the polytope

$$\{ x \in [0,1]^E : \sum_{e \in E} x_e = n - 1, \sum_{e \in \delta(X)} x_e \geq 1 \text{ for } \emptyset \neq X \subset V \}.$$ (4 Points)

Deadline: Tuesday, December 1, 2015, before the lecture.

Information: Submissions by groups of up to three students are allowed.