Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 2

Exercise 2.1: Prove: An undirected graph G is 2-edge-connected if and only if $|E(G)| \geq 2$ and G has an ear-decomposition. (2 Points)

Exercise 2.2: Suppose that two workers have to carry out a number of jobs. Both workers need 1 hour for each job, and there are certain jobs that need to be done before certain other jobs. The task is to get all jobs done as early as possible.
This can be modeled as an acyclic directed graph $G = (V, E)$ where an edge $e = (i, j)$ means that job i has do be finished before job j is started.
Let $E' := \{\{i, j\} \subseteq V \mid \text{there is neither an } i\text{-}j\text{-path nor a } j\text{-}i\text{-path in } G\}$ and set $H := (V, E')$. Prove:

(i) The workers cannot finish their work after less than $|V| - \nu(H)$ hours. (1 Points)

(ii) The workers can finish their work after $|V| - \nu(H)$ hours. (3 Points)

Exercise 2.3: Prove: A graph G has a perfect matching if and only if for each $X \subseteq V(G)$, the graph $G - X$ has at most $|X|$ factor-critical components. (3 Points)

Exercise 2.4: Prove: Every 3-regular simple graph with at most two bridges has a perfect matching. Find an example for a 3-regular simple graph that has no perfect matching. (3 Points)

Exercise 2.5: Let $G = (V, E)$ be a bipartite graph with bipartition

$$V = \{a_1, \ldots, a_k\} \dot\cup \{b_1, \ldots, b_k\}.$$

For any vector $x = (x_e)_{e \in E}$, we define a matrix $M_G(x) = (m_{ij}^x)_{1 \leq i,j \leq k}$ by

$$m_{ij}^x = \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Its determinant $\det M_G(x)$ is a polynomial in x. We further define the permanent of a $k \times k$ matrix M as

$$\operatorname{per}(M) := \sum_{\pi \in S_k} \prod_{i=1}^{k} m_{i\pi(i)},$$

where S_k is the set of permutations of $\{1, \ldots, k\}$. Prove:

(i) G has a perfect matching if and only if $\det M_G(x)$ is not identically 0. (2 Points)

(ii) If G is simple, it has exactly $\operatorname{per}(M_G(1, \ldots, 1))$ perfect matchings. (2 Points)

Deadline: Tuesday, November 10, 2015, **before** the lecture.
Information: Submissions by groups of up to **three** students are allowed.