Winter term 2015/16
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Combinatorial Optimization

Exercise Sheet 1

Exercise 1.1: Find an infinite counterexample to Hall's Theorem. More precisely: Find a bipartite graph $G = (A \dot{\cup} B, E)$ with $A \cong \mathbb{N}$, $B \cong \mathbb{N}$, and $|\Gamma(S)| \geq |S|$ for every $S \subseteq A$ and every $S \subseteq B$ such that G does not contain a perfect matching.
(4 Points)

Exercise 1.2:

1. Let M_1 and M_2 be two (inclusion-wise) maximal matchings in a graph G. Prove that $|M_1| \leq 2|M_2|$. (2 Points)
2. Let G be a bipartite graph such that for each proper subset $F \subsetneq E(G)$ and $G' := (V(G), F)$ we have $\tau(G') < \tau(G)$. Prove that $E(G)$ is a matching.
(2 Points)

Exercise 1.3: Let G be a bipartite graph. For each $v \in V(G)$, let $<_v$ be a linear ordering of $\delta(v)$. Prove that there is a matching $M \subsetneq E(G)$ such that for each $e \in E(G) \setminus M$ there is an edge $f \in M$ and a vertex $v \in V(G)$ such that $v \in (e \cap f)$ and $e <_v f$. (4 Points)

Exercise 1.4:
Let G be a graph. Prove following equalities:

1. $\alpha(G) + \tau(G) = |V(G)|$ for any graph G. (1 Points)
2. $\nu(G) + \zeta(G) = |V(G)|$ for any graph G with no isolated vertices. (2 Points)
3. $\zeta(G) = \alpha(G)$ for any bipartite graph G with no isolated vertices. (1 Points)

Deadline: Tuesday, November 3, 2015, before the lecture.
Information: Submissions by groups of one or two students are allowed.