Combinatorial Optimization

Exercise Sheet 1

Exercise 1.1: Find an infinite counterexample to Hall’s Theorem. More precisely: Find a bipartite graph $G = (A \cup B, E)$ with $A \cong \mathbb{N}$, $B \cong \mathbb{N}$, and $|\Gamma(S)| \geq |S|$ for every $S \subseteq A$ and every $S \subseteq B$ such that $G$ does not contain a perfect matching.

(4 Points)

Exercise 1.2:

1. Let $M_1$ and $M_2$ be two (inclusion-wise) maximal matchings in a graph $G$. Prove that $|M_1| \leq 2|M_2|$.

(2 Points)

2. Let $G$ be a bipartite graph such that for each proper subset $F \subsetneq E(G)$ and $G' := (V(G), F)$ we have $\tau(G') < \tau(G)$. Prove that $E(G)$ is a matching.

(2 Points)

Exercise 1.3: Let $G$ be a bipartite graph. For each $v \in V(G)$, let $<_v$ be a linear ordering of $\delta(v)$. Prove that there is a matching $M \subsetneq E(G)$ such that for each $e \in E(G) \setminus M$ there is an edge $f \in M$ and a vertex $v \in V(G)$ such that $v \in (e \cap f)$ and $e <_v f$.

(4 Points)

Exercise 1.4:

Let $G$ be a graph. Prove following equalities:

1. $\alpha(G) + \tau(G) = |V(G)|$ for any graph $G$.

(1 Points)

2. $\nu(G) + \zeta(G) = |V(G)|$ for any graph $G$ with no isolated vertices.

(2 Points)

3. $\zeta(G) = \alpha(G)$ for any bipartite graph $G$ with no isolated vertices.

(1 Points)

Deadline: Tuesday, November 3, 2015, before the lecture.

Information: Submissions by groups of one or two students are allowed.