Combinatorial Optimization

Exercise Sheet 2

Exercise 2.1:
Prove that an undirected graph $G$ is factor-critical if and only if $G$ is connected and $\nu(G) = \nu(G - v)$ for all $v \in V(G)$.

(4 points)

Exercise 2.2:
Let $G = (V, E)$ be a graph. A set $\mathcal{H} = \{S_1, \ldots, S_k, v_1, \ldots v_r\}$ has property A if

- $S_i \subseteq V$ and $|S_i|$ is odd for $1 \leq i \leq k$,
- $v_i \in V$ for $1 \leq i \leq r$, and
- for each $e \in E$ either $e \subseteq S_i$ for some $i \in \{1, \ldots, k\}$ or $v_i \in e$ for some $i \in \{1, \ldots, r\}$.

The weight of a set $\mathcal{H}$ with property A is $w(\mathcal{H}) := r + \sum_{i=1}^{k} \frac{|S_i| - 1}{2}$. Prove $\nu(G) = \min \{w(\mathcal{H}) | \mathcal{H} \text{ is a set with property A}\}$.

(4 points)

Exercise 2.3:
Let $G = (V, E)$ be a graph and $X \subseteq V$. Let $\beta(G, X)$ be the maximum size of a set $Y \subseteq X$ for which there is a matching in $G$ that covers $Y$. Prove $\beta(G, X) = \min_{U \subseteq V} |X| + |U| - q_X(U)$.

Here $q_X(U)$ denotes the number of odd connected components of $G - U$ whose vertices are all in $X$.

Hint: Construct a new graph by adding vertices $V'$ with $|V'| = |V|$ and all edges between vertices in $V'$. For some $X' \subseteq V'$ with $|X'| = |X| - \beta(G, X)$ add the edges $E(X', V)$ and $E(V' \setminus X', V \setminus X)$.

(4 points)
Exercise 2.4:
Let $G = (V, E)$ be a $k$-vertex-connected graph with $2\nu(G) < |V| - 1$.

- Prove $\nu(G) \geq k$.
- Prove $\tau(G) \leq 2\nu(G) - k$.

(4 points)

Deadline: Thursday, October 31, 2013, before the lecture.