Exercise 1:
Let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$, $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$. For any vector $x = (x_e)_{e \in E(G)}$ we define a matrix $M_G(x) = (m_{ij}^x)_{1 \leq i,j \leq k}$ by
\[
m_{ij}^x := \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}
\]
Its determinant $\det M_G(x)$ is a polynomial in $x = (x_e)_{e \in E(G)}$. Prove that $G$ has a perfect matching if and only if $\det M_G(x)$ is not identically zero. (4 points)

Exercise 2:
Let $G$ be a graph and $M$ a matching in $G$ that is not maximum.
(a) Show that there are $\nu(G) - |M|$ vertex-disjoint $M$-augmenting paths in $G$.
   
   Hint: Recall the proof of Berge’s Theorem (Thm. 6).

(b) Prove that there exists an $M$-augmenting path of length at most $\frac{\nu(G) + |M|}{\nu(G) - |M|}$.

(c) Let $P$ be a shortest $M$-augmenting path in $G$, and $P'$ an $(M \triangle E(P))$-augmenting path. Then $|E(P')| \geq |E(P)| + |E(P \cap P')|$.

Consider the following generic algorithm. We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen. By (c), $|E(P_k)| \leq |E(P_{k+1})|$ for all $k$.

(d) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$ then $P_i$ and $P_j$ are vertex-disjoint.

(e) Conclude that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains at most $2\sqrt{\nu(G)} + 2$ different numbers. (6 points)

Exercise 3:
For a graph $G$, let $T(G) := \{X \subseteq V(G)|q_g(X) > |X|\}$ the family of Tutte-sets of $G$. Prove or find a counterexample: $G$ is factor-critical if and only if $T(G) = \{\emptyset\}$. (4 points)

Exercise 4: Prove:
1. An undirected graph $G$ is 2-edge-connected if and only if $|E(G)| \geq 2$ and $G$ has an ear-decomposition.

2. A directed graph is strongly connected if and only if it has an ear-decomposition. (4 points)

Deadline: Tuesday, November 2nd, before the lecture.