Summer term 2026
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Linear and Integer Optimization

Exercise Sheet 5

Exercise 5.1: (Dual Simplex Algorithm)
Let $\max\{c^\intercal x : Ax = b, x \geq 0\}$ be an LP that is not unbounded and where A has full row rank. Consider the following algorithm that has as an input the LP and a dual feasible basis B, i.e. $z_N = c_N - A_N^\intercal A_B^{-\intercal} c_B \leq 0$.

BTRAN:
Solve $A_B x_B = b$

PRICING:
If $x_B \geq 0$, **Stop**. Else choose an $i \in \{1, \ldots, m\}$ with $x_{B_i} < 0$.

FTRAN:
Solve $A_B^\intercal w = e_i$ and compute $\alpha_N = A_N^\intercal w$.

RATIO-Test:
If $\alpha_N \geq 0$, **Stop**. Else choose a

$$j = \arg\min\left\{\frac{z_k}{\alpha_k} : \alpha_k < 0, k \in N\right\}, \text{ and set } \gamma = \frac{z_j}{\alpha_j}.$$

Update:

$$\begin{aligned} z_N &\leftarrow z_N - \gamma\alpha_N, & z_{B_i} &\leftarrow -\gamma, \\ N &\leftarrow N \setminus \{j\} \cup \{B_i\}, & B_i &\leftarrow j \quad (\text{now } B \leftarrow B \setminus \{B_i\} \cup \{j\}). \end{aligned}$$

Goto **BTRAN**.

Prove

1. If the algorithm stops in the PRICING step, then B is an optimum basis and x_B with $x_N = 0$ is an optimum basic solution.
2. If the algorithm stops in the RATIO-Test, then $P^=(A, b) = \emptyset$.
3. The UPDATE-step transforms a dual feasible basis B into a new basis that is dual feasible again. (2+2+3 points)

Exercise 5.2: Consider the following linear program

$$\begin{array}{rrcrcrcrcrcrcl} \max & 9x_1 & + & 3x_2 & + & x_3 & & & & & & & & \\ \text{s.t.} & x_1 & & & & & + & x_4 & & & & & = & 1 \\ & 6x_1 & + & x_2 & & & & & + & x_5 & & & = & 9 \\ & 18x_1 & + & 6x_2 & + & x_3 & & & & & + & x_6 & = & 81 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

Show that the simplex algorithm traverses all vertices of the polyhedron when using $\{4, 5, 6\}$ as a starting basis, and choosing $j \in \arg\max\{z_j \ : \ j \in N\}$ in the pricing step. (4 points)

Exercise 5.3: Consider a linear program $\max\{c^\intercal x \mid Ax = b, x \geq 0\}$. Let B be a feasible basis with basic solution x^* and reduced cost vector $z \leq 0$ (so x^* is an optimum solution). Let $I = \{j \in N \mid z_j = 0\}$.

(a) Prove that x^* is the unique optimum solution if $I = \emptyset$.

(b) Assume that $I \neq \emptyset$. Prove that in this case x^* is the unique optimum solution if and only if the following linear program has the optimum solution value 0:

$$\begin{array}{lrcll} \max & \sum_{i \in I} x_i & & & \\ \text{s.t.} & Ax & = & b & \\ & x_i & = & 0 & \text{for } i \in N \setminus I \\ & x_i & \geq & 0 & \text{for } i \in B \cup I \end{array}$$

(2+3 points)

Exercise 5.4: In the job assignment problem, n jobs with execution times $t_1, ..., t_n \in \mathbb{R}_{\geq 0}$ need to be processed by m workers. For each job i we are given by $S_i \subseteq \{1, ..., m\}$ the set of workers that are qualified to perform job i. It is possible for several workers to process the same job in parallel to speed up the process but one worker can only process one job at a time.

(a) Formulate an LP minimizing the *makespan* for processing all jobs (the time until the last worker finishes).

(b) Dualize this LP. (2+2 points)

Submission deadline: Tuesday, May 19, 2026, 16:00, via eCampus (in groups of at most 3 students).