Prof. Dr. Vera Traub
Jannis Blauth

Exercise Set 10

Exercise 10.1. Show that the analysis of the primal-dual algorithm for the STEINER FOREST PROBLEM is tight (i.e., the approximation guarantee of the primal-dual algorithm is not better than 2).

(4 points)

Exercise 10.2. Show that the clean-up step at the end of the primal-dual algorithm for the STEINER FOREST PROBLEM is crucial: without the clean-up step, the algorithm does not even achieve any finite performance ratio.

(4 points)

Exercise 10.3. Let $G = (V, E)$ be a directed graph with root $r \in V$ and nonnegative edge weights c. Consider the following linear programming relaxation for the MINIMUM WEIGHT ARBORESCENCE PROBLEM:

$$\begin{array}{llll} \min & c^\top x & & \\ & \sum_{e \in \delta^-(S)} x_e & \geq 1 & \forall\, S \subseteq V(G) \setminus \{r\} \\ & x & \geq 0\,. & \end{array}$$

Consider the following primal-dual algorithm for the MINIMUM WEIGHT ARBORESCENCE PROBLEM.

Input: A directed graph $G = (V, E)$ with root $r \in V$ and nonnegative edge weights c.

1. Set $y_S := 0$ for each $S \subseteq V \setminus \{r\}$ and $F := \emptyset$.
2. Set $i := 0$.
3. Let $\mathcal{A}$ denote the family of minimal vertex sets $A \subseteq V(G) \setminus \{r\}$ with $\delta^-_F(A) = 0$.
4. While $\mathcal{A} \neq \emptyset$ do:
5. Set $i := i + 1$.
6. Increase y_A for each $A \in \mathcal{A}$ uniformly until $c_e = \sum_{S: e \in \delta^-(S)} y_S$ for some $A' \in \mathcal{A}$ and $e \in \delta^-(A')$. Let e_i denote this edge.
7. Set $F = F \cup \{e_i\}$.
8. Update $\mathcal{A}$ as in Step 3.
9. For $j := i$ down to 1 do:
10. If $F \setminus \{e_j\}$ satisfies $\delta^-_F(S) \geq 1$ for all $S \subseteq V(G) \setminus \{r\}$, set $F := F \setminus \{e_j\}$.
11. Return F.

Show that this algorithm computes an arborescence and a dual solution y to the above LP of the same cost.

(5 points)

Exercise 10.4. An instance of the PRIZE-COLLECTING STEINER FOREST PROBLEM consists of an instance of the STEINER FOREST PROBLEM plus a penalty $\pi_{\{v,w\}} \in \mathbb{R}_+$ for each terminal pair $\{v, w\}$. The goal is to find a spanning forest H which minimizes $c(E(H)) + \pi(H)$, where $\pi(H)$ is the sum of penalties of terminal pairs that are not in the same connected component of H. The natural LP relaxation is:

$$\begin{array}{llll} \min & c^\top x + \pi^\top z & & \\ & \sum_{e \in \delta(U)} x_e + z_{\{v,w\}} & \geq 1 & \forall v \in U \subset V(G) \setminus \{w\} \\ & x, z & \geq 0 \,. & \end{array}$$

Consider the following threshold rounding approach: Let (x, z) be an optimum solution to the above LP. For some $0 \leq \alpha < 1$, set $x'_e := \frac{1}{1-\alpha} \cdot x_e$ for all e; then x' is a feasible solution to the LP relaxation of the STEINER FOREST PROBLEM from the lecture restricted to the terminal pairs $\{v, w\}$ for which $z_{\{v,w\}} \leq \alpha$.

(i) Show that for $\alpha = \frac{1}{3}$ this yields a 3-approximation.

(ii) Show that by trying different values for α, one can obtain a $\frac{1}{1-e^{-\frac{1}{2}}}$-approximation.

Hint: First, choose α with respect to some distribution. Then, derandomize the algorithm.

(2+5 points)

Deadline: Tuesday, June 25$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss24/appr_ss24_ex.html`

In case of any questions feel free to contact me at blauth@or.uni-bonn.de.