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Exercise Set 4

Exercise 4.1. Consider the following greedy algorithm for the KNAPSACK PROBLEM: Sort the indices such that $\frac{c_1}{w_1} \geq \cdots \geq \frac{c_n}{w_n}$, and set $S := \emptyset$. For $i := 1$ to n do: If $\sum_{j \in S \cup \{i\}} w_j \leq W$, then set S to $S \cup \{i\}$.

Prove that there is no constant k such that this is a k-approximation algorithm.

(2 points)

Exercise 4.2.

(a) Consider the FRACTIONAL MULTI KNAPSACK PROBLEM: Given natural numbers $n, m \in \mathbb{N}$ and w_i, $c_{ij} \in \mathbb{N}$ as well as $W_j \in \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, find $x_{ij} \geq 0$ satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} x_{ij} w_i \leq W_j$ for all $1 \leq j \leq m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum (or decide that no such x_{ij} exist).

State a polynomial-time combinatorial algorithm for this problem.
(Do not use that a linear program can be solved in polynomial time.)

(b) Can we solve the integral MULTI KNAPSACK PROBLEM (i.e. $x_{ij} \in \{0, 1\}$) in pseudo-polynomial time if m is fixed?

(4+2 points)

Exercise 4.3. The KNAPSACK PROBLEM can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \; : \; \sum_{i=1}^{n} w_i x_i \leq W, \, x_i \in \{0, 1\} \, \forall \, 1 \leq i \leq n \right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LP}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0, 1\}$ is replaced by $0 \leq x_i \leq 1$.

Show that the *integrality gap*

$$\sup_{\mathcal{I}} \left\{ \frac{\mathrm{LP}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \; : \; \mathrm{OPT}(\mathcal{I}) \neq 0 \right\}$$

of the KNAPSACK PROBLEM is unbounded. What is the integrality gap of the KNAPSACK PROBLEM restricted to instances with $w_i \leq W$ for all $i = 1, \ldots, n$?

(3 points)

Exercise 4.4. Show that the following variant of the KNAPSACK PROBLEM is NP-hard:

$$\max\left\{\sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \leq W,\, x_i \in \mathbb{Z}_{\geq 0}\,\forall\, 1 \leq i \leq n\right\} \tag{2}$$

(Here, we allow to use an item several times.) You may use that the KNAPSACK PROBLEM is NP-hard.

(5 points)

Exercise 4.5. Recall the version of KNAPSACK from Exercise 4.4, where items can be used multiple times.

Give an FPTAS for this problem.

(4 points)

Deadline: Tuesday, May 7^{th}, until 2:15 PM (before the lecture) on paper or per upload on eCampus. Solutions may be submitted in groups of up to 2 people.

The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss24/appr_ss24_ex.html`

In case of any questions feel free to contact me at puhlmann@or.uni-bonn.de.