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Exercise Set 2

Exercise 2.1. For $k \in \mathbb{N}$ consider the following problem:

Instance: A set U and a set $\mathcal{S}$ of subsets of U with $|S| \leq k$ for all $S \in \mathcal{S}$, weights $w : U \to \mathbb{R}_{\geq 0}$.

Task: Find $T \subseteq U$ such that $T \cap S \neq \emptyset$ for each $S \in \mathcal{S}$ and $\sum_{t \in T} w(t)$ minimum.

(i) Show that this problem is NP-hard for $k \geq 2$.

(ii) Give a polynomial time k-factor approximation algorithm.

(iii) Give a linear time k-factor approximation algorithm for the special case that $w(t) = 1$ for $t \in U$.

(1+2+2 points)

Exercise 2.2. Consider the standard IP formulation of the Minimim Weight Set Cover Problem, and its LP-relaxation

$$\min \left\{ cx \ : \ \sum_{S \in \mathcal{S} : e \in S} x_S \geq 1 \text{ for all } e \in U,\ x_S \geq 0 \text{ for all } S \in \mathcal{S} \right\}.$$

Consider the algorithm that picks all sets associated with non-zero values in an optimum solution to the LP-relaxation. Show that this algorithm achieves an approximation guarantee of p if each element $e \in U$ is contained in at most p sets.

(3 points)

Exercise 2.3. Consider the following variant of Set Cover:

Instance: A set U, sets $\mathcal{S} = \{S_1, \ldots, S_m\}$ such that $\bigcup_{S \in \mathcal{S}} S = U$, an integer $k \in \mathbb{N}$.

Output: k sets $S_{i_1}, \ldots, S_{i_k} \in \mathcal{S}$ such that $\left| \bigcup_{j=1}^{k} S_{i_j} \right|$ is maximum.

Show that iteratively picking the element that maximizes the amount of not yet covered elements is a $(1 - \frac{1}{e})$-approximation.

(4 points)

Exercise 2.4. Prove that Satisfiability remains NP-complete if each clause contains at most three literals and each variable occurs in at most three clauses.

(3 points)

Exercise 2.5. An instance of MAX-SAT is called k-satisfiable if any k of its clauses can be satisfied simultaneously. Following the hint, develop a polynomial-time algorithm that computes for every 2-satisfiable instance a truth assignment which satisfies at least a $\frac{\sqrt{5}-1}{2}$-fraction of the clauses.

Hint: Some variables occur in one-element clauses (w.l.o.g. all one-element clauses are positive), set them *true* with probability a (for some constant $a \in [0, 1]$), and set the other variables *true* with probability $\frac{1}{2}$. Choose a appropriately and derandomize this algorithm.

(5 points)

Deadline: Tuesday, April 23$^{\text{rd}}$, until 2:15 PM (before the lecture) on paper or per upload on eCampus. Solutions may be submitted in groups of up to 2 people.

The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss24/appr_ss24_ex.html`

In case of any questions feel free to contact me at puhlmann@or.uni-bonn.de.