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Exercise Set 11

Exercise 11.1. Let $G = (A \dot\cup B, E)$ be a bipartite graph. Assume that there is a matching covering A. Let $\varepsilon > 0$. Use the Resource Sharing Algorithm to find variables $(x_e)_{e\in E} \in [0,1]^{E(G)}$ that satisfy

$$\sum_{e\in\delta(v)} x_e = 1 \qquad \forall v \in A, \qquad \sum_{e\in\delta(w)} x_e \leq 1+\varepsilon \qquad \forall w \in B$$

within a running time of $\mathcal{O}(|E|\frac{\ln|B|}{\varepsilon^2})$.

(5 points)

Exercise 11.2. Let $\delta, m > 0$. Let X be an instance of the Resource Sharing Problem with 1 customer, m resources and $\lambda^* = 1$ such that the Resource Sharing Algorithm requires at least t phases to compute a $(1+\delta)$-approximation for X.

Show that there exists an instance of the Resource Sharing Problem with 2 customers, $2m$ resources and $\lambda^* = 1$ such that the Resource Sharing Algorithm requires at least tm oracle calls to compute a $(1+\delta)$-approximation.

Hint: First construct an instance with 1 customer and m resources that requires m oracle calls in a single phase.

(5 points)

Exercise 11.3. Prove that the number of oracle calls after $t \in \mathbb{N}$ phases of the core Resource Sharing Algorithm is bounded by

$$t|\mathcal{N}| + \frac{|\mathcal{R}|}{\epsilon} \ln \frac{||y^{(t)}||_1}{|\mathcal{R}|}.$$

Hint: Proceed similarly to the proof of Lemma 5.11 in the lecture notes.

(5 points)

Exercise 11.4. In this exercise we use the notation from Section 5.3.5 from the lecture notes. Let $y \in \mathbb{R}^{\mathcal{R}}_{>0}$ be a price vector. Consider one arrival time customer v. Recall that there is an arrival time solution $a(v) \in$

$\{a_{\min}(v), a_{max}(v)\}$ that minimizes

$$f(t) = \sum_{r=(v,w)\in\delta^+(v)} y_r \frac{t - a_{\min}(v)}{a_{\max}(w) - a_{\min}(v)} + \sum_{r=(u,v)\in\delta^-(v)} y_r \frac{a_{\max}(v) - t}{a_{\max}(v) - a_{\min}(u)}$$

in $[a_{\min}(v), a_{\max}(v)]$.

Let $\epsilon > 0$. Consider the following algorithm to iteratively compute an arrival time solution.

1: **for** $i = 1, \dots, n$ **do**
2: Compute $a_i(v) \in \{a_{\min}(v), a_{\max}(v)\}$ minimizing $f(a_i(v))$;
3: $y_r \leftarrow y_r \cdot e^{\frac{\epsilon}{n} \cdot \mathrm{usg}_{v,r}(a_i(v))}$ for all $r \in \mathcal{R}$;
4: **end for**
5: **return** $a(v) \leftarrow \frac{1}{n}\sum_{i=1}^{n} a_i(v)$;

Prove that for $n \to \infty$ the output of the above algorithm converges to $\min\{\max\{a_{\min}(v), t^*\}, a_{\max}(v)\}$, where t^* is the unique root of the function

$$g(t) = \sum_{r=(v,w)\in\delta^+(v)} \frac{y_r}{a_{\max}(w) - a_{\min}(v)} \cdot e^{\epsilon \cdot \frac{t - a_{\min}(v)}{a_{\max}(w) - a_{\min}(v)}} - \sum_{r=(u,v)\in\delta^-(v)} \frac{y_r}{a_{\max}(v) - a_{\min}(u)} \cdot e^{\epsilon \cdot \frac{a_{\max}(v) - t}{a_{\max}(v) - a_{\min}(u)}} .$$

(5 points)

Deadline: June 27, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss23/chipss23_ex.html`

In case of any questions feel free to contact me at drees@or.uni-bonn.de.