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Exercise Set 11

Exercise 11.1. Show that in a slight variant of JAIN'S ALGORITHM the number of iterations in which we have to solve an LP can be bounded by

(a) $2|V(G)|^2$

(b) $2|V(G)|$.

Here we set $x_e := x_e + \lfloor y_e \rfloor$ for all e if some $y_e \geq 1$, otherwise we update x as before.

Hint: Conclude from Lemma 20.38 that in the second case all but $2|V(G)| - 2$ *edges can be deleted. For (b), delete one more edge in each iteration.*

(5 points)

Exercise 11.2. An instance of the MINIMUM BOUNDED DEGREE SPANNING TREE PROBLEM (MDBSTP) is an undirected graph G with non-negative edge weights c and degree constraints $B_v \geq 1$ for every vertex v. The task is to find a spanning tree T fulfilling all degree constraints, i.e. with $|\delta_T(v)| \leq B_v$ for every vertex $v \in V(G)$, that is cheapest possible. The goal of this exercise is to develop an algorithm that computes a spanning tree T of cost at most OPT that violates every degree costraint by at most one.

We consider the following LP relaxation for the MDBSTP (where $W = V(G)$):

$$
\begin{aligned}
\min \sum_{e \in E(G)} c(e) x_e & & \\
\text{s.t.} \quad \sum_{e \in E(G)} x_e &= |V| - 1 & \\
\sum_{e \in E(G[S])} x_e &\leq |S| - 1 & (\emptyset \neq S \subseteq V(G)) \\
\sum_{e \in \delta(v)} x_e &\leq B_v & (v \in W) \\
x_e &\geq 0 & (e \in E(G))
\end{aligned}
\tag{1}
$$

(a) Let x^* be an extreme point solution of (1) for some set $W \subseteq V(G)$ and assume $x^*_e > 0$ for all $e \in E(G)$. Call a nonempty vertex set S tight if $x^*(E(G[S])) = |S| - 1$. Prove that there exists a laminar family $\mathcal{L}$ of tight sets such that $\{\chi^{E(G[S])} : S \in \mathcal{L}\}$ is a basis of the span of $\{\chi^{E(G[S])} : S \text{ tight}\}$.

We will assume in the following that the above LP relaxation has a feasible solution. (Otherwise, no feasible solution exists.) Now consider the following algorithm:

1. $W \leftarrow V(G)$

2. While $W \neq \emptyset$:

- Find an optimal extreme point solution x^* to (1) and remove every edge e with $x^*_e = 0$ from G. Let E^* be the support of x^* .
- If there exists a vertex $v \in W$ such that $|\delta_{E^*}(v)| \leq B_v + 1$, then remove v from W.

3. Find an optimal extreme point solution x^* and return the support of x^*.

(b) Suppose the above algorithm terminates. Show that then it returns the edge set of a a spanning tree of cost at most OPT that violates every degree constraint by at most one.

We now prove that the above algorithm terminates. Suppose $W \neq \emptyset$ and we do not remove any vertex from W because $|\delta_{E^*}(v)| \geq B_v + 2$ for every $v \in W$.

Let $X \subseteq W$ be the vertices with tight degree constraints (i.e. with $x^*(\delta(v)) = B_v$) and $\mathcal{L}$ as in (a). Note that $|E^*| \leq |X| + |\mathcal{L}|$. To derive a contradiction we use a counting argument. We give one token to each edge $e \in E^*$ and redistribute these tokens (fractionally) to the vertices of G and the sets in $\mathcal{L}$. The redistribution works as follows. Every edge $e \in E^*$ gives an x^*_e fraction of its token to the minimal set $S \in \mathcal{L}$ such that both endpoints of e are contained in S. Moreover, it gives a $\frac{1}{2}(1 - x^*_e)$ fraction of its token to each of its endpoints.

(c) Prove that every set $S \in \mathcal{L}$ and every vertex $v \in X$ receives at least one token.

(d) Prove:

- $V(G) \in \mathcal{L}$
- For every vertex $v \in V(G) \setminus X$ and every edge $e \in \delta(v)$ we have $x^*_e = 1$.
- For every edge e with $x^*_e = 1$, the incidence vector χ^e is contained in the span of $\{\chi^S : S \in \mathcal{L}\}$.
- The vectors $\{\chi^{\delta(v)} : v \in X\} \cup \{\chi^{E(G[S])} : S \in \mathcal{L}\}$ are linearly independent.

(e) Use (d) to derive a contradiction.

(5+1+3+4+2 points)

Submission: You can submit your solutions in groups of 2 people, either on paper in the lecture or via upload on Sciebo (link on website, late submissions after 2.15 pm will not be considered).

Deadline: Tuesday, June 27$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/lectures/ss23/ss23.html`

In case of any questions feel free to contact me at ellerbrock@or.uni-bonn.de.