Professor Dr. Vera Traub
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Exercise Set 10

Exercise 10.1. Consider the 2-Edge-Connected Spanning Subgraph Problem as the special case of Survivable Network Design where $r_{xy} = 2$ for all $x, y \in V(G)$. Show that there exists a combinatorial 2-approximation algorithm for this problem.

Hint: You may use that there exists a polynomial time algorithm solving the following problem: Given a directed graph $G = (V, E)$, $k \in \mathbb{N}$, a root $r \in V$ and edge weigths $c : E \to \mathbb{R}$, find a minimum-weight subgraph H of G such that H contains k edge-disjoint directed r-v-paths for each $v \in V$.

(5 points)

Exercise 10.2. An instance of Prize-Collecting Steiner Forest consists of an instance of Steiner Forest plus a penalty $\pi_{\{v,w\}} \in \mathbb{R}_+$ for each terminal pair $\{v, w\}$. The goal is to find a spanning forest H which minimizes $c(E(H)) + \pi(H)$, where $\pi(H)$ is the sum of penalties of terminal pairs that are not in the same connected component of H. The natural LP relaxation is:

$$\begin{aligned} \min\ & c^\top x + \pi^\top z \\ \text{s.t.} \sum_{e \in \delta(U)} & x_e + z_{\{v,w\}} \geq 1 \qquad \forall\ v \in U \subset V(G) \setminus \{w\} \\ & x, z \geq 0 \end{aligned}$$

Consider the following threshold rounding approach: Let (x, z) be an optimum solution to the above LP. For some $0 \leq \alpha < 1$, set $x'_e := \frac{1}{1-\alpha} \cdot x_e$ for all e; then x' is a feasible solution to the LP relaxation of the Steiner Forest Problem from the lecture restricted to the terminal pairs $\{v, w\}$ for which $z_{\{v,w\}} \leq \alpha$.

(a) Show that for $\alpha = \frac{1}{3}$ this yields a 3-factor approximation algorithm for Prize-Collecting Steiner Forest.

(b) Show that by choosing α uniformly in $[0, \gamma]$ for a good choice of $\gamma \in (0, 1)$, one can obtain a randomized $\frac{1}{1-e^{-1/2}}$-factor approximation algorithm. (Note that it is possible to derandomize but you do not need to do that here.)

(2+3 points)

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Exercise 10.3. Consider the POINT-TO-POINT CONNECTION PROBLEM: Given an undirected graph G with edge weights $c : E(G) \to \mathbb{R}_+$ and sets $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ and $|S| = |T| \geq 1$, find a set $F \subseteq E(G)$ of minimum cost such that there is a bijection $\pi : S \to T$ and paths from s to $\pi(s)$ for all $s \in S$ in $(V(G), F)$.

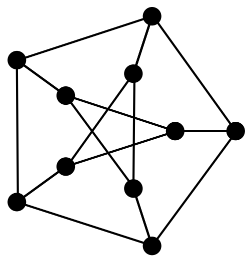
Show that $f(X) = 1$ if $|X \cap S| \neq |X \cap T|$ and $f(X) = 0$ otherwise defines a proper function.

(5 points)

Exercise 10.4.

$$\begin{aligned} \min \sum_{e \in E(G)} x_e & \\ \text{s.t.} \sum_{e \in \delta(S)} x_e \geq f(S) & \quad (S \subseteq V(G)) \\ x_e \geq 0 & \quad (e \in E(G)) \end{aligned}$$

Find an optimum basic solution x for the above linear program, where G is the Petersen graph (see figure below) and $f(S) = 1$ for all $\emptyset \neq S \subsetneq V(G)$. Find a maximal laminar family $\mathcal{B}$ of tight sets with respect to x such that the vectors $\chi^{\delta(B)}$, $B \in \mathcal{B}$, are linearly independent.



(5 points)

Submission: You can submit your solutions in groups of 2 people, either on paper in the lecture or via upload on Sciebo (link on website, late submissions after 2.15 pm will not be considered).

Deadline: Tuesday, June 20$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/lectures/ss23/ss23.html`

In case of any questions feel free to contact me at ellerbrock@or.uni-bonn.de.