Professor Dr. Vera Traub
Antonia Ellerbrock

Exercise Set 8

Exercise 8.1. Let $(U, \mathcal{S}, c)$ be an instance of Minimum Weight Set Cover and $p := \max_{S\in\mathcal{S}} |S|$. Assume w.l.o.g. that for every set $S \in \mathcal{S}$ also all its subsets $R \subseteq S$ are present in $\mathcal{S}$ and we have $c(R) \leq c(S)$. Consider the following algorithm:

1. Let $\mathcal{R}$ be an arbitrary set cover solution, where w.l.o.g. $\mathcal{R}$ is a partition of U.
2. Do the following as long as it decreases the potential function $\Phi(\mathcal{R}) := \sum_{R\in\mathcal{R}} H_{|R|} \cdot c(R)$:
   - For each set $R \in \mathcal{R}$, define $\overline{c}(u) := \frac{1}{|R|} \cdot c(R)$ for all $u \in R$.
   - Choose $S \in \mathcal{S}$ that minimizes $H_{|S|} \cdot c(S) - \sum_{u\in S} \overline{c}(u)$.
   - Replace every set $R \in \mathcal{R}$ by $R \setminus S$.
   - Add S to $\mathcal{R}$.
3. Return $\mathcal{R}$.

(a) Prove that the above algorithm terminates in finite time.

(b) Prove that the output $\mathcal{R}$ of the above algorithm satisfies $c(\mathcal{R}) \leq H_p \cdot \mathrm{OPT}$, where OPT denotes the value of an optimum solution.

(c) How can the above algorithm be modified to run in polynomial time while losing only an arbitrarily small error of $\varepsilon > 0$ in the approximation ratio?

(1+4+2 points)

Exercise 8.2. Consider an undirected graph $G = (V, E)$ with non-negative edge weights $c : E \to \mathbb{R}_+$ and terminal set $T \subseteq V$. The Bottleneck Steiner Tree Problem asks for a Steiner tree S for T in G whose bottleneck weight $\max_{e\in E(S)} c(e)$ is minimum.

Show that Bottleneck Steiner Tree can be solved in polynomial time.

(3 points)

Exercise 8.3. Consider an instance $(G, c : E(G) \to \mathbb{R}_+, T \subseteq V(G))$ of Steiner Tree and let $r \in T$ be an arbitrarily chosen root. Let

$$\text{LP} = \min \left\{ c(x) : \sum_{e \in \delta(U)} x_e \geq 1 \text{ for } U \subseteq V(G) \setminus \{r\} \text{ with } U \cap T \neq \emptyset, x \geq 0 \right\}.$$

Now we replace every edge $\{v, w\}$ by two directed edges (v, w) and (w, v) (with cost $c(\{v, w\})$). Let

$$\text{BCR} = \min \left\{ c(x) : \sum_{e \in \delta^-(U)} x_e \geq 1 \text{ for } U \subseteq V(G) \setminus \{r\} \text{ with } U \cap T \neq \emptyset, x \geq 0 \right\}.$$

(a) Prove that the value BCR is independent of the choice of the root $r \in T$.

(b) What is the supremum of $\frac{\text{BCR}}{\text{LP}}$ over all instances (with $\text{LP} \neq 0$)?

(3+2 points)

Exercise 8.4. Let $G = (V, E)$ be an undirected graph. For a partition $\mathcal{P}$ of the vertex set V, let

$$\delta(\mathcal{P}) := \{e \in E : e \in \delta(U) \text{ for some } U \in \mathcal{P}\}.$$

Prove:

$$\left\{ x \in [0,1]^E : \sum_{e \in E} x_e = |V(G)| - 1, \sum_{e \in E(G[X])} x_e \leq |X| - 1 \text{ for } \emptyset \neq X \subsetneq V(G) \right\}$$

$$= \left\{ x \in [0,1]^E : \sum_{e \in E} x_e = |V(G)| - 1, \sum_{e \in \delta(\mathcal{P})} x_e \geq |\mathcal{P}| - 1 \text{ for every partition } \mathcal{P} \text{ of } V \right\}$$

(5 points)

Submission: You can submit your solutions in groups of 2 people, either on paper in the lecture or via upload on Sciebo (link on website, late submissions after 2.15 pm will not be considered).

Deadline: Tuesday, June 6th, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/lectures/ss23/ss23.html`

In case of any questions feel free to contact me at ellerbrock@or.uni-bonn.de.