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Exercise Set 3

Exercise 3.1. Consider the following local search algorithm for the (unweighted) Maximum Cut problem: Start with an arbitrary vertex set $S \subseteq V$. Iterate the following: If a single vertex can be added to S or can be removed from S such that $|\delta(S)|$ increases, do so. If no such vertex exists, terminate and return $\delta(S)$.

(a) Prove that this algorithm is a 2-approximation algorithm. (In particular, show that it runs in polynomial time.)

(b) Find an example that proves that the analysis is tight, even if we start with $S = \emptyset$.

(c) Does the algorithm always find an optimum solution for planar graphs or bipartite graphs?

(d) Give a linear-time 2-approximation algorithm for the Maximum Cut problem in graphs with nonnegative edge weights.

(e) Is your analysis tight? If yes, provide a suitable example with $|V(G)| = 3$.

(f) In the Directed Maximum Weight Cut Problem we are given a digraph G with weights $c : E(G) \to \mathbb{R}_+$ and we are searching for a set $X \subseteq V(G)$ that maximizes $\sum_{e \in \delta^+(X)} c(e)$. Show that there is a 4-approximation algorithm for this problem.

(2+2+2+2+1+2 points)

Exercise 3.2. The Knapsack Problem can be formulated as the following integer program:

$$\max\left\{\sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \le W,\, x_i \in \{0,1\}\,\forall\, 1 \le i \le n\right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LP}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0,1\}$ is replaced by $0 \le x_i \le 1$.

Show that the *integrality gap*

$$\sup_{\mathcal{I}}\left\{\frac{\mathrm{LP}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \,:\, \mathrm{OPT}(\mathcal{I}) \ne 0\right\}$$

of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with $w_i \le W$ for all $i = 1, \ldots, n$?

(3 points)

Exercise 3.3. (a) Consider the Fractional Multi Knapsack Problem: Given natural numbers $n, m \in \mathbb{N}$ and w_i, $c_{ij} \in \mathbb{N}$ as well as $W_j \in \mathbb{N}$ for $1 \le i \le n$ and $1 \le j \le m$, find x_{ij} satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \le i \le n$ and $\sum_{i=1}^{n} x_{ij} w_i \le W_j$ for all $1 \le j \le m$ such that $\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum.

State a polynomial-time combinatorial algorithm for this problem.
(Do not use that a linear program can be solved in polynomial time.)

(b) Can we solve the integral Multi Knapsack Problem in pseudopolynomial time if m is fixed?

(4+2 points)

Submission: You can submit your solutions in groups of 2 people, either on paper in the lecture or via upload on Sciebo to

`https://uni-bonn.sciebo.de/s/omVU1VMioEQwDa0`

(late submissions after 2.15 pm will not be considered).

Deadline: Tuesday, April 25$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/lectures/ss23/ss23.html`

In case of any questions feel free to contact me at ellerbrock@or.uni-bonn.de.