Exercise Set 11

Recall again the LPs DCR from Exercise 10.1 and BCR from Exercise 9.1. You may use in the following that the vertices of BCR are integral if all vertices are terminals (i.e. $V = R$ where V is the vertex set and R the set of terminals).

Given a Steiner tree instance $G = (V, E)$ with edge costs c and terminals R, let $R' \subseteq R$ be a subset of the terminals with $|R'| = k$. Let S be a Steiner tree in this instance. Define the set $\mathrm{Br}_S(R')$ of *bridges* of S with respect to R' to be the $k-1$ most expensive edges that can be removed from S after contracting R' (while keeping connectivity of S).

Exercise 11.1. Define $w\colon R^2 \to \mathbb{R}$ as mapping each pair (s, t) of terminals to the cost of the most expensive edge of the unique s-t-path in S (i.e. $w(s, t) = c(\mathrm{Br}_S(\{s, t\}))$.) Show: For any subset $R' \subseteq R$ there is a tree Y in the terminal distance graph $G_D(R')$ such that

(a) Y spans R' and

(b) $w(Y) = c(\mathrm{Br}_S(R'))$ and

(c) for any edge $\{s, t\} \in Y$, the s-t-path in S contains exactly one edge from $\mathrm{Br}_S(R')$.

(5 points)

Exercise 11.2. Let x denote a vector that is an optimum solution to DCR and let $\mathcal{C}$ be the set of components C for which $x_C > 0$. Let T be a spanning tree in the terminal distance graph $G_D(R)$.

(a) Construct a tree Y_C for each component $C \in \mathcal{C}$ as in Exercise 11.1, choose a root for each Y_C and direct the trees Y_C. Define $y\colon R^2 \to \mathbb{R}$ by $y(u, v) := \sum_{Y_C:(u,v)\in E(Y_C)} x_C$. Show that there is a spanning tree F in $G_D(R)$ with $w(F) \le \sum_{(u,v)\in R^2} w(e)y(e)$.

(b) Prove: $w(F) \ge c(T)$.

(c) Show:

$$c(T) \le \sum_{C\in\mathcal{C}} x_C \cdot c(\mathrm{Br}_T(C)).$$

(3+2+2 points)

Exercise 11.3. Given a Steiner tree instance $G = (V, E)$ with edge costs c and terminals R, denote by $\mathrm{OPT}_{\mathrm{DCR}}$ the value of an optimum solution to DCR. Let T be a minimum-cost spanning tree in the terminal distance graph. Show: $c(T) \leq 2\,\mathrm{OPT}_{\mathrm{DCR}}$.

(4 points)

Deadline: June 28th, until 2:15 PM (before the lecture) via eCampus. LaTeX-submissions are highly encouraged, however, you can also submit a scan (e.g. obtained with a mobile phone). Solutions may be submitted in groups of up to 2 people.

The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/appr_ss22_ex.html`

In case of any questions feel free to contact me at puhlmann@or.uni-bonn.de.