Exercise Set 9

Exercise 9.1. Let $G = (V, E)$ with edge costs $c\colon E \to \mathbb{R}_{\geq 0}$ with terminal set T denote an instance of the Steiner Tree Problem and let $r \in T$ be an arbitrarily chosen root. Let

$$\mathrm{LP} = \min\left\{\sum_{e \in E} c(e)x_e : \sum_{e \in \delta(U)} x_e \geq 1 \text{ for } U \subseteq V \setminus \{r\} \text{ with } U \cap T \neq \emptyset, x \geq 0\right\}.$$

Now we replace every edge $\{v, w\}$ by two directed edges (v, w) and (w, v) (with cost $c(\{v, w\})$). Call the edge set arising in this way $\overleftrightarrow{E}$. Consider the following LP (the so-called bidirected cut relaxation):

$$\mathrm{BCR} = \min\left\{\sum_{e \in \overleftrightarrow{E}} c(e)x_e : \sum_{e \in \delta^-(U)} x_e \geq 1 \text{ for } U \subseteq V \setminus \{r\} \text{ with } U \cap T \neq \emptyset, x \geq 0\right\}.$$

(a) Prove that the value BCR is independent of the choice of the root $r \in T$.

(b) What is the supremum of $\frac{\mathrm{BCR}}{\mathrm{LP}}$ over all instances (with $\mathrm{LP} \neq 0$)?

(4 points)

Exercise 9.2. Show that the contraction lemma still holds when the edges added between terminals have lengths larger than 0. (We add parallel edges if there already is an edge.)

(3 points)

Exercise 9.3. Show that the "vertex version" of the contraction lemma is wrong:

Construct a complete graph with metric edge lengths and vertex sets A, B and C, such that

$$0 < \mathrm{mst}(A) - \mathrm{mst}(A \cup C) < \mathrm{mst}(A \cup B) - \mathrm{mst}(A \cup B \cup C),$$

where $\mathrm{mst}(X)$ for a vertex set X denotes the length of a minimum spanning tree in the graph induced by X.

(3 points)

Definition. Let $G = (V, E), c\colon E \to \mathbb{R}_+$ and $R \subseteq V$ be an instance of the Steiner tree problem. A *directed full Steiner tree* is defined as follows: Take a full Steiner tree, choose one terminal as a *sink* and direct all edges of the Steiner tree towards the sink. All other terminals of this directed full component are called *sources*.

Exercise 9.4. Denote by $\mathcal{C}_k = \{C_1, C_2, \dots\}$ a set of minimum directed full Steiner trees in G for all subsets of R with at most k terminals. $C \in \mathcal{C}_k$ *crosses* $U \subseteq R$ if C has at least one source in U and the sink in $R \setminus U$. Define $\delta_k^+(U) := \{C \in \mathcal{C}_k : C \text{ crosses } U\}$. Choose an arbitrary terminal $r \in R$ as a *root* and consider the *directed cut relaxation*:

$$\begin{array}{lll} & \min & \displaystyle\sum_j c(C_j) \cdot x_j \\ k\text{-DCR} & \text{s.t.} & \displaystyle\sum_{C_j \in \delta_k^+(U)} x_j \geq 1 \quad \forall \emptyset \subsetneq U \subseteq R \setminus \{r\}, \\ & & x_j \geq 0 \quad \forall j \end{array} \tag{1}$$

Show that the LP k-DCR can be solved in polynomial time for any fixed k.

Hint: Prove that one can find in polynomial time a set U with $\emptyset \subsetneq U \subseteq R \setminus \{r\}$ that minimizes $\sum_{C_j \in \delta_k^+(U)} x_j$.

(6 points)

Deadline: Tuesday, June 14th, until 2:15 PM (before the lecture) via eCampus. LaTeX-submissions are highly encouraged, however, you can also submit a scan (e.g. obtained with a mobile phone). Solutions may be submitted in groups of up to 2 people.

The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/appr_ss22_ex.html`

In case of any questions feel free to contact me at puhlmann@or.uni-bonn.de.