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Exercise Set 5

Exercise 5.1.

(a) Consider the FRACTIONAL MULTI KNAPSACK PROBLEM: Given natural numbers $n, m \in \mathbb{N}$ and w_i, $c_{ij} \in \mathbb{N}$ as well as $W_j \in \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, find $x_{ij} \geq 0$ satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} x_{ij} w_i \leq W_j$ for all $1 \leq j \leq m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum (or decide that no such x_{ij} exist).

State a polynomial-time combinatorial algorithm for this problem.
(Do not use that a linear program can be solved in polynomial time.)

(b) Can we solve the integral MULTI KNAPSACK PROBLEM (i.e. $x_{ij} \in \{0, 1\}$) in pseudo-polynomial time if m is fixed?

(4+4 points)

Exercise 5.2. The KNAPSACK PROBLEM can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \leq W,\, x_i \in \{0,1\}\, \forall\, 1 \leq i \leq n \right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LP}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0, 1\}$ is replaced by $0 \leq x_i \leq 1$.

Show that the *integrality gap*

$$\sup_{\mathcal{I}} \left\{ \frac{\mathrm{LP}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \,:\, \mathrm{OPT}(\mathcal{I}) \neq 0 \right\}$$

of the KNAPSACK PROBLEM is unbounded. What is the integrality gap of the KNAPSACK PROBLEM restricted to instances with $w_i \leq W$ for all $i = 1, \ldots, n$?

(3 points)

Exercise 5.3. Show that the following variant of the KNAPSACK PROBLEM is NP-hard:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \leq W,\, x_i \in \mathbb{Z}_{\geq 0}\, \forall\, 1 \leq i \leq n \right\} \tag{2}$$

(Here, we allow to use an item several times.) You may use that the KNAPSACK PROBLEM is NP-hard.

(5 points)

Deadline: Tuesday, May 10th, until 2:15 PM (before the lecture) via eCampus. LaTeX-submissions are highly encouraged, however, you can also submit a scan (e.g. obtained with a mobile phone). Solutions may be submitted in groups of up to 2 people.

The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/appr_ss22_ex.html`

In case of any questions feel free to contact me at puhlmann@or.uni-bonn.de.