# Exercise Set 7

**Exercise 7.1.** Let $(G, c, T)$ be an instance of the Steiner Tree Problem, $G$ connected, $t \in T$ a terminal and $k \in \mathbb{N}$ with $k \geq 1$.

(a) For two valid lower bounds $\text{lb}_a$ and $\text{lb}_b$, define $\max(\text{lb}_a, \text{lb}_b)$ by

$$\max(\text{lb}_a, \text{lb}_b)(v, I) := \max\Big(\text{lb}_a(v, I), \text{lb}_b(v, I)\Big).$$

Show that $\max(\text{lb}_a, \text{lb}_b)$ also defines a valid lower bound.

(b) Prove that $\text{lb}_{\text{BB}}(v, I) := \text{BB}(\{v\} \cup I)$ is a valid lower bound for instances of the Rectilinear Steiner Tree Problem.

(c) Show that $\text{lb}_{\text{mst}}(v, I) := \frac{\text{mst}(\{v\} \cup I)}{2}$ defines a valid lower bound. Here $\text{mst}(\{v\} \cup I)$ denotes the cost of a minimal spanning tree in $G'[\{v\} \cup I]$, where $G'$ is the metric closure of $G$.

(3 + 1 + 1 points)

**Exercise 7.2.** Let $T$ be an instance of the Rectilinear Steiner Tree Problem and $r \in T$. For a rectilinear Steiner tree $Y$ we denote by $f(Y)$ the maximum length of a path from $r$ to any element of $T \setminus \{r\}$ in $Y$.

(a) Find an instance where no Steiner tree minimizes both length and $f$.

(b) Consider the problem of finding a shortest Steiner tree $Y$ minimizing $f(Y)$ among all shortest Steiner trees. Is there always a tree with these properties which is a subgraph of the Hanan grid?

(1 + 4 points)

**Exercise 7.3.** For a finite set $\emptyset \neq T \subsetneq \mathbb{R}^2$ we define

$$\text{BB}(T) := \max_{(x,y) \in T} x - \min_{(x,y) \in T} x + \max_{(x,y) \in T} y - \min_{(x,y) \in T} y.$$

A *Steiner tree* for $T$ is a tree $Y$ with $T \subseteq V(Y) \subsetneq \mathbb{R}^2$. We denote by $\text{Steiner}(T)$ the length of a shortest rectilinear (i.e. edge lengths acc. to $\ell_1$) Steiner tree for $T$. Moreover let $\text{MST}(T)$ be the length of a minimum spanning tree in the complete graph on $T$ with edge costs $\ell_1$.

Prove that:

(a) $\mathrm{BB}(T) \leq \mathrm{Steiner}(T) \leq \mathrm{MST}(T)$;

(b) $\mathrm{Steiner}(T) \leq \frac{3}{2}\,\mathrm{BB}(T)$ for $|T| \leq 5$;

(c) There is no $\alpha \in \mathbb{R}$ s.t. $\mathrm{Steiner}(T) \leq \alpha\,\mathrm{BB}(T)$ for all finite $\emptyset \neq T \subset \mathbb{R}^2$.

(1 + 2 + 2 points)

**Exercise 7.4.** Let $Y$ be a Steiner tree for terminal set $T$ with $|T| \geq 2$ in which all leaves are terminals. Prove

$$\sum_{t \in T} \Big( |\delta_Y(t)| - 1 \Big) = k - 1$$

where $k$ is the number of full components of $Y$.

(5 points)

**Deadline:** June 11th, 12:15. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss20/chipss20_ex.html`

In case of any questions feel free to contact me at ahrens@dm.uni-bonn.de.