1. In the job assignment problem, \( n \) jobs with execution times \( t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \) need to be processed by \( m \) workers. For each job \( i \) we are given by \( S_i \subseteq \{1, \ldots, m\} \) the set of workers that are qualified to perform job \( i \). It is possible for several workers to process the same job in parallel to speed up the process but one worker can only process one job at a time.

(a) Formulate an LP minimizing the makespan for processing all jobs (the time until the last worker finishes).

(b) Dualize this LP. (2+2 points)

2. Prove that a polyhedron \( P \subseteq \mathbb{R}^n \) is of dimension \( n \) if and only if \( P \) contains a vector \( x \) in its interior (i.e. there is some \( \varepsilon > 0 \) such that an \( n \)-dimensional ball \( B = \{y \in \mathbb{R}^n \mid \|y - x\|_2 \leq \varepsilon\} \) with radius \( \varepsilon \) and center \( x \) is contained in \( P \)). (4 points)

3. Let \( C_n := [-1,+1]^n \) be an \( n \)-dimensional hypercube. Determine the number \( f_k \) of \( k \)-dimensional faces for \( k = 0 \ldots n \) and the total number of faces. Prove the correctness of your answers. (3 points)

4. For a polytope \( P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset \) let \( P' := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid Ax \leq tb, 0 \leq t \leq 1\} \).

(a) Show that \( P' = \text{conv}(\{(P \times \{1\}) \cup \{0\})\).\)

(b) Prove that for each face \( F \) of \( P \) the set \( \text{conv}((F \times \{1\}) \cup \{0\}) \) is a face of \( P' \).

(c) Do these these statements still necessarily hold if \( P \) is an unbounded polyhedron? (2+2+1 points)

5. Prove that any set \( X \subseteq \mathbb{R}^n \) with \( |X| > n + 1 \) can be decomposed into subsets \( X_1 \) and \( X_2 \) such that \( \text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset \). (4 points)

Due date: Thursday, May 2, 2019, before the lecture.