Exercise Set 9

Exercise 9.1. Consider the ESCAPE ROUTING PROBLEM: We are given a complete 2-dimensional grid graph $G = (V, E)$ (i.e. $V = \{0, \dots, k-1\} \times \{0, \dots, k-1\}$ and $E = \{\{v, w\} \mid v, w \in V, \|v - w\| = 1\}$) and a set $P = \{p_1, \dots, p_m\} \subseteq V$. The task is to compute vertex-disjoint paths $\{q_1, \dots, q_m\}$ s.t. each q_i connects p_i with a point on the border $B = \{(x, y) \in V \mid \{x, y\} \cap \{0, k-1\} \neq \emptyset\}$.

Find a polynomial-time algorithm for the ESCAPE ROUTING PROBLEM or prove that the problem is NP-hard.

(4 points)

Exercise 9.2. Let (G, H) be a pair of undirected graphs on $V(G) = V(H)$ with capacities $u : E(G) \to \mathbb{R}_+$ and demands $b : E(H) \to \mathbb{R}_+$. A *concurrent flow* of value $\alpha > 0$ is a family $(x^f)_{f \in E(H)}$ where x^f is an s-t-flow of value $\alpha \cdot b(f)$ in $(V(G), \{(v, w), (w, v) \mid \{v, w\} \in E(G)\})$ for each $f = \{t, s\} \in E(H)$, and

$$\sum_{f \in E(H)} x^f\Big((v, w)\Big) + x^f\Big((w, v)\Big) \leq u(e)$$

for all $e = \{v, w\} \in E(G)$. The MAXIMUM CONCURRENT FLOW PROBLEM is to find a concurrent flow with maximum value $\alpha > 0$.

Prove that the MAXIMUM CONCURRENT FLOW PROBLEM is a special case of the MIN-MAX RESOURCE SHARING PROBLEM. Specify how to implement block solvers.

(5 points)

Exercise 9.3. In this exercise we want to show that the randomized rounding for the MIN-MAX RESOURCE SHARING PROBLEM can be derandomized. Let $\mathcal{N} = \{1, ..., |\mathcal{N}|\}$ and let $\mathcal{B}_i \subseteq \mathbb{R}^{\mathcal{R}}$ $(i \in \mathcal{N})$ be finite sets. Given an instance of the MIN-MAX RESOURCE SHARING PROBLEM, let $(x_{i,b})_{i \in \mathcal{N}, b \in \mathcal{B}_i}$ be a fractional solution with $\sum_{b \in \mathcal{B}_i} x_{i,b} = 1$ for all $i \in \mathcal{N}$. For $z_1 \in \mathcal{B}_1, ..., z_l \in \mathcal{B}_l$ let $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l)$ denote the probability that for a randomized rounding $(\hat{z}_i)_{i \in \mathcal{N}}$ of x (choosing $\hat{z}_i = b$ with probability $x_{i,b}$) we have $\hat{\lambda} > (1+\delta)\lambda$ under the condition that $\hat{z}_1 = z_1, ..., \hat{z}_l = z_l$. Let $\rho_r := \max\{b_r | i \in \mathcal{N}, b \in \mathcal{B}_i, r \in \mathcal{R}\}$. We will use the following algorithm, known as METHOD OF CONDITIONAL PROBABILITIES, to round x.

```
for i = 1, ..., |N| do
    Set ẑ_i := b where b ∈ B_i minimizes Pr(λ̂ > (1 + δ)λ | ẑ_1, ..., ẑ_{i−1}, b)
end for
```

(a) If $\Pr(\hat{\lambda} > (1+\delta)\lambda) < 1$, show that the method of conditional probabilities returns a rounding $\hat{z}$ satisfying $\hat{\lambda} \leq (1+\delta)\lambda$.

(b) Let $F_\delta(z_1, ..., z_l) :=$

$$\sum_{r \in \mathcal{R}} \prod_{i=1,...,l} ((1+\delta)^{(z_i)_r/\rho_r}) \prod_{i=l+1}^{|\mathcal{N}|} (\sum_{b \in \mathcal{B}_i} x_{i,b}(1+\delta)^{b_r/\rho_r})(1+\delta)^{-(1+\delta)\lambda/\rho_r}$$

Show that $F_\delta(z_1, ..., z_l)$ is a pessimistic estimator for $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l)$, i.e. show that the following two statements hold:

- $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l) \leq F_\delta(z_1, ..., z_l)$
- $\min_{b \in \mathcal{B}_i} F_\delta(z_1, ..., z_{i-1}, b) \leq F_\delta(z_1, ..., z_{i-1})$

(c) Assume that

$$1 - \sum_{r \in \mathcal{R}} e^{-((1+\delta)\ln(1+\delta)-\delta)\lambda/\rho_r} > 0$$

Show that $F_\delta < 1$ and that the method of conditional probabilities returns a solution $\hat{z}$ satisfying $\hat{\lambda} \leq (1+\delta)\lambda$ when minimizing $F_\delta(\hat{z}_1, ..., \hat{z}_l, b)$ instead of $\Pr(\hat{\lambda} > (1+\delta)\lambda | \hat{z}_1, ..., \hat{z}_l, b)$.

(4+3+3 points)

The exercise class on June 7th will be held in the lecture room.

Deadline: Tuesday, June 11th, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ss19/chipss19.html

In case of any questions feel free to contact me at klotz@or.uni-bonn.de.