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Exercise Set 4

Exercise 4.1. For a finite set $\emptyset \neq T \subsetneq \mathbb{R}^2$ we define

$$\mathrm{BB}(T) := \max_{(x,y)\in T} x - \min_{(x,y)\in T} x + \max_{(x,y)\in T} y - \min_{(x,y)\in T} y.$$

A *Steiner tree* for T is a tree Y with $T \subseteq V(Y) \subsetneq \mathbb{R}^2$. We denote by $\mathrm{Steiner}(T)$ the length of a shortest rectilinear (i.e. edge lengths acc. to ℓ_1) Steiner tree for T. Moreover let $\mathrm{MST}(T)$ be the length of a minimum spanning tree in the complete graph on T with edge costs ℓ_1.

Prove that:

(a) $\mathrm{BB}(T) \leq \mathrm{Steiner}(T) \leq \mathrm{MST}(T)$;

(b) $\mathrm{Steiner}(T) \leq \frac{3}{2}\,\mathrm{BB}(T)$ for $|T| \leq 5$;

(c) There is no $\alpha \in \mathbb{R}$ s.t. $\mathrm{Steiner}(T) \leq \alpha\,\mathrm{BB}(T)$ for all finite $\emptyset \neq T \subset \mathbb{R}^2$.

(2 + 3 + 2 points)

Exercise 4.2. (a) Show that the Steiner ratio for the Euclidean plane is at least $2/\sqrt{3}$.

(b) Show that the Steiner ratio for a metric space is at most $2 - 2/n$ where n is the number of terminals.

(2 + 3 points)

Exercise 4.3. Consider the following algorithm to compute a rectilinear Steiner tree T for a set P of points in the plane $\mathbb{R}^2$.

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Choose p ∈ P arbitrarily;
T := ({p}, ∅), S := P \ {p}
while S ≠ ∅ do
    Choose s ∈ S with minimum dist(s, T)
    Let {u, w} ∈ E(T) be an edge which minimizes dist(s, SP(u, w))
    v := arg min{dist(s, v) | v ∈ SP(u, w)}
    T := (V(T) ∪ {v} ∪ {s}, E(T) \ (u, w) ∪ {u, v} ∪ {v, w} ∪ {v, s}}
    S := S \ {s}
end while
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In this notation $SP(u,w) \subset \mathbb{R}^2$ is the area covered by shortest paths between u and w, and $dist(s,T)$ is the minimum distance between s and the shortest path area $SP(u,w)$ of an edge $\{u,w\} \in E(T)$.

Show that the algorithm is a $\frac{3}{2}$-approximation algorithm for the Minimum Steiner Tree Problem.

Hint: First show that the length of T is at most the length of a minimum spanning tree on P.

(8 points)

Deadline: Thursday, May 2$^{\text{nd}}$, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ss19/chipss19.html

In case of any questions feel free to contact me at klotz@or.uni-bonn.de.