Exercise Set 9

Exercise 9.1. Let \((G, H)\) be a pair of undirected graphs on \(V(G) = V(H)\) with capacities \(u : E(G) \to \mathbb{R}_+\) and demands \(b : E(H) \to \mathbb{R}_+\). A concurrent flow of value \(\alpha > 0\) is a family \((x^f)_{f \in E(H)}\) where \(x^f\) is an \(s-t\)-flow of value \(\alpha \cdot b(f)\) in \((V(G), \{(v, w), (w, v) \mid \{v, w\} \in E(G)\})\) for each \(f = \{t, s\} \in E(H)\), and
\[
\sum_{f \in E(H)} x^f(\{v, w\}) + x^f(\{w, v\}) \leq u(e)
\]
for all \(e = \{v, w\} \in E(G)\). The Maximum Concurrent Flow Problem is to find a concurrent flow with maximum value \(\alpha > 0\).

Prove that the Maximum Concurrent Flow Problem is a special case of the Min-Max Resource Sharing Problem. Specify how to implement block solvers.

(5 points)

Exercise 9.2. Consider the Escape Routing Problem: We are given a complete 2-dimensional grid graph \(G = (V, E)\) (i.e. \(V = \{0, \ldots, k - 1\} \times \{0, \ldots, k - 1\}\) and \(E = \{(v, w) \mid v, w \in V, \|v - w\| = 1\}\) and a set \(P = \{p_1, \ldots, p_m\} \subseteq V\). The task is to compute vertex-disjoint paths \(\{q_1, \ldots, q_m\}\) s.t. each \(q_i\) connects \(p_i\) with a point on the border \(B = \{(x, y) \in V \mid \{x, y\} \cap \{0, k - 1\} \neq \emptyset\}\).

Find a polynomial-time algorithm for the Escape Routing Problem or prove that the problem is NP-hard.

(4 points)

Exercise 9.3. Show that the Vertex-Disjoint Paths Problem is NP-complete even if \(G\) is a subgraph of a track graph \(G_T\) with two routing planes. Recall that in this case \(G_T\) is a graph \(G_T = (V, E)\) for some \(n_x, n_y \in \mathbb{N}\) with \(V = \{1, \ldots, n_x\} \times \{1, \ldots, n_y\} \times \{1, 2\}\) and \(E = \{(x, y, z), (x', y', z') \mid |x - x'| + |y - y'| + |z - z'| = 1\}\).

\textbf{Hint:} Consider the proof of Theorem 5.2.

(5 points)
Exercise 9.4. Given an instance of the **Min-Max Resource Sharing Problem** with \( \sigma \)-optimal block solvers for some fixed \( \sigma \geq 1 \).

(a) Show that \( t \) phases of the **Resource Sharing Algorithm** call the oracle at most

\[
\min \left\{ t \Lambda, \ t |N| + \frac{|R'|}{\varepsilon} \ln \left( \prod y^{(t)} \right) \right\}
\]

times where \( \Lambda := \sum_{N \in N} \max \{1, \sup \{b_r \mid r \in R, b \in B_N\} \} \) and \( R' := \{r \in R \mid \exists N \in N, b \in B_N \text{ with } b_r > 1\} \).

(b) Prove that a \( \sigma(1 + \omega) \)-approximate solution can be computed in

\[
O\left(\theta \log |R| \left( (|N| + |R|) \log \log |R| + \sigma \omega^{-2} \min \{ \rho |N|, |N| + |R| \} \right) \right)
\]

time where \( \rho := \max\{1, \sup \{b_r/\lambda^* \mid r \in R, N \in N, b \in B_N\} \} \) and \( \overline{R} := \{r \in R \mid \exists N \in N, b \in B_N \text{ with } b_r > \lambda^*\} \).

Remark: For practical routing instances \( \rho \) and \( |\overline{R}| \) are usually small.

(2 + 4 points)

**Deadline:** June 26\(^{th}\), before the lecture. The websites for lecture and exercises can be found at:

[http://www.or.uni-bonn.de/lectures/ss18/chipss18.html](http://www.or.uni-bonn.de/lectures/ss18/chipss18.html)

In case of any questions feel free to contact me at bihler@or.uni-bonn.de.

**Save the date:** The student council of mathematics will organize the math party on 21/06 in the N8schicht. The presale will be held on Mon 18/06, Tue 19/06 and Wed 20/06 in the mensa Poppelsdorf. Further information can be found at [http://fsmath.uni-bonn.de/](http://fsmath.uni-bonn.de/)