Exercise Set 4

Exercise 4.1. For a finite set \( \emptyset \neq T \subseteq \mathbb{R}^2 \) we define

\[
BB(T) := \max_{(x, y) \in T} x - \min_{(x, y) \in T} x + \max_{(x, y) \in T} y - \min_{(x, y) \in T} y.
\]

A Steiner tree for \( T \) is a tree \( Y \) with \( T \subseteq V(Y) \subseteq \mathbb{R}^2 \). We denote by \( \text{Steiner}(T) \) the length of a shortest rectilinear (i.e. edge lengths acc. to \( \ell_1 \)) Steiner tree for \( T \). Moreover let \( \text{MST}(T) \) be the length of a minimum spanning tree in the complete graph on \( T \) with edge costs \( \ell_1 \).

Prove that:

(a) \( BB(T) \leq \text{Steiner}(T) \leq \text{MST}(T) \);

(b) \( \text{Steiner}(T) \leq \frac{3}{2} BB(T) \) for \( |T| \leq 5 \);

(c) There is no \( \alpha \in \mathbb{R} \) s.t. \( \text{Steiner}(T) \leq \alpha BB(T) \) for all finite \( \emptyset \neq T \subset \mathbb{R}^2 \).

(2 + 3 + 2 points)

Exercise 4.2. Let \( T \) be an instance of the Rectilinear Steiner Tree Problem and \( r \in T \). For a rectilinear Steiner tree \( Y \) we denote by \( f(Y) \) the maximum length of a path from \( r \) to any element of \( T \setminus \{r\} \) in \( Y \).

(a) Find an instance where no Steiner tree minimizes both length and \( f \).

(b) Consider the problem of finding a shortest Steiner tree \( Y \) minimizing \( f(Y) \) among all shortest Steiner trees. Is there always a tree with these properties which is a subgraph of the Hanan grid?

(1 + 4 points)

Exercise 4.3. Consider the following algorithm to compute a rectilinear Steiner tree \( T \) for a set \( P \) of points in the plane \( \mathbb{R}^2 \).

In this notation \( SP(u, w) \subset \mathbb{R}^2 \) is the area covered by shortest paths between \( u \) and \( w \), and \( \text{dist}(s, T) \) is the minimum distance between \( s \) and the shortest path area \( SP(u, w) \) of an edge \( \{u, w\} \in E(T) \).
1: Choose $p \in P$ arbitrarily;
2: $T := (\{p\}, \emptyset), S := P \setminus \{p\}$
3: while $S \neq \emptyset$ do
4: Choose $s \in S$ with minimum $\text{dist}(s, T)$
5: Let $\{u, w\} \in E(T)$ be an edge which minimizes $\text{dist}(s, SP(u, w))$
6: $v := \arg \min \{\text{dist}(s, v) \mid v \in SP(u, w)\}$
7: $T := (V(T) \cup \{v\} \cup \{s\}, E(T) \setminus \{u, w\} \cup \{u, v\} \cup \{v, w\} \cup \{v, s\})$
8: $S := S \setminus \{s\}$
9: end while

Show that the algorithm is a $\frac{3}{2}$-approximation algorithm for the Minimum Steiner Tree Problem.

Hint: First show that the length of $T$ is at most the length of a minimum spanning tree on $P$.

(8 points)

Deadline: Tuesday, May 15th, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ss18/chipss18.html

In case of any questions feel free to contact me at bihler@or.uni-bonn.de.