Exercise Set 8

Exercise 8.1. Show that in a slight variant of JAIN'S ALGORITHM the number of iterations in which we have to solve an LP can be bounded by

(a) $2|V(G)|^2$

(b) $2|V(G)|$.

Here we set $x_e := x_e + \lfloor y_e \rfloor$ for all e if some $y_e \geq 1$, otherwise we update x as before.

Hint: Conclude from Lemma 20.38 that in the second case all but $|V(G)|-2$ edges can be deleted. For (b), delete one more edge in each iteration.

(5 points)

Exercise 8.2. An instance of the MINIMUM BOUNDED DEGREE SPANNING TREE PROBLEM (MDBSTP) is an undirected graph G with non-negative edge weights c and degree constraints $B_v \geq 1$ for every vertex v. The task is to find a spanning tree T fulfilling all degree constraints, i.e. with $|\delta_T(v)| \leq B_v$ for every vertex $v \in V(G)$, that is cheapest possible. The goal of this exercise is to develop an algorithm that computes a spanning tree T of cost at most OPT that violates every degree costraint by at most one.

We consider the following LP relaxation for the MDBSTP (where $W = V(G)$)

$$\begin{aligned}
\min \sum_{e \in E(G)} c(e) x_e & & \\
\text{s.t.} \quad \sum_{e \in E(G)} x_e &= |V| - 1 & \\
\sum_{e \in E(G[S])} x_e &\leq |S| - 1 & (\emptyset \neq S \subseteq V(G)) \\
\sum_{e \in \delta(v)} x_e &\leq B_v & (v \in W) \\
x_e &\geq 0 & (e \in E(G)).
\end{aligned} \tag{1}$$

(a) Let x^* be an extreme point solution of (1) for some set $W \subseteq V(G)$. Call a nonempty vertex set S tight if $x^*(E(G[S])) = |S| - 1$. Prove that there exists a laminar family $\mathcal{L}$ of tight sets such that $\{\chi^{E(G[S])} : S \in \mathcal{L}\}$ is a basis of the span of $\{\chi^{E(G[S])} : S \text{ tight}\}$.

We will assume in the following that the above LP relaxation has a feasible solution. (Otherwise, no feasible solution exists.) Now consider the following algorithm:

1. $W \leftarrow V(G)$
2. While $W \neq \emptyset$:
   - Find an optimal extreme point solution x^* to (1) and remove every edge e with $x^*_e = 0$ from G. Let E^* be the support of x^* .
   - If there exists a vertex $v \in W$ such that $|\delta_{E^*}(v)| \leq B_v + 1$ then remove v from W.
3. Find an optimal extreme point solution x^* and return the support of x^*.

(b) Suppose the above algorithm terminates. Show that then it returns the edge set of a a spanning tree of cost at most OPT that violates every degree costraint by at most one.

We now prove that the above algorithm terminates. Suppose $W \neq \emptyset$ and we do not remove any vertex from W because $|\delta_{E^*}(v)| \geq B_v + 2$ for every $v \in W$.

Let $X \subseteq W$ be the vertices with tight degree constraints (i.e. with $x^*(\delta(v)) = B_v$) and $\mathcal{L}$ as in (a). Note that $|E^*| \leq |X| + |\mathcal{L}|$. To derive a contradiction we use a counting argument. We give one token to each edge $e \in E^*$ and redistribute these tokens (fractionally) to the vertices of G and the sets in $\mathcal{L}$. The redistribution works as follows. Every edge $e \in E^*$ gives an x^*_e fraction of its token to the minimal set $S \in \mathcal{L}$ such that both endpoints of e are contained in S. Moreover, it gives a $\frac{1}{2}(1 - x^*_e)$ fraction of its token to each of its endpoints.

(c) Prove that every set $S \in \mathcal{L}$ and every vertex $v \in X$ receives at least one token.

(d) Prove:
   - $V(G) \in \mathcal{L}$
   - For every vertex $v \in V(G) \setminus X$ and every edge $e \in \delta(v)$ we have $x^*_e = 1$.

- For every edge e with $x_e^* = 1$, the incidence vector χ^e is contained in the span of $\{\chi^S : S \in \mathcal{L}\}$.
- The vectors $\{\chi^{\delta(v)} : v \in X\} \cup \{\chi^{E(G[S])} : S \in \mathcal{L}\}$ are linearly independent.

(e) Use (d) to derive a contradiction.

(15 points)

Announcement by the student council: The student council of mathematics will organize the math party on 21/06 in N8schicht. The presale will be held on Mon 18/06, Tue 19/06 and Wed 20/06 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de

Deadline: Thursday, June 21$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss18/appr_ss18_ex.html`

In case of any questions feel free to contact me at traub@or.uni-bonn.de.