Exercise Set 3

Exercise 3.1. Consider the following local search algorithm for the Maximum Cut problem: Start with an arbitrary vertex set $S \subseteq V$. Iterate the following: If a single vertex can be added to $S$ or can be removed from $S$ such that $|\delta(S)|$ increases, do so. If no such vertex exists, terminate and return $\delta(S)$.

(a) Prove that this algorithm is a 2-approximation algorithm. (In particular, show that it runs in polynomial time.)

(b) Does the algorithm always find an optimum solution for planar graphs or bipartite graphs?

(c) Give a linear time 2-approximation algorithm for the Maximum Cut problem in graphs with nonnegative edge weights.

(6 points)

Exercise 3.2. Describe exact algorithms with running times of $O(2^n)$ for the following problems:

(a) Subset Sum:
Given $K, n, x_1, \ldots, x_n \in \mathbb{N}$, find $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = K$ (or decide that no such set exists).

(b) Knapsack (where $n$ denotes the number of items).

(5 points)

Exercise 3.3. The Knapsack Problem can be formulated as integer program:

$$
\max \left\{ \sum_{i=1}^{n} c_i x_i : \sum_{i=1}^{n} w_i x_i \leq W, x_i \in \{0, 1\} \forall 1 \leq i \leq n \right\}
$$

(1)

For an instance $I$, denote the optimum of (1) by $\text{OPT}(I)$ and let $\text{LP}(I)$ be the optimum of the linear relaxation, where $x_i \in \{0, 1\}$ is replaced by $0 \leq x_i \leq 1$.

Show that the integrality gap

$$
\sup_I \left\{ \frac{\text{LP}(I)}{\text{OPT}(I)} : \text{OPT}(I) \neq 0 \right\}
$$
of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with \( w_i \leq W \) for all \( i = 1, \ldots, n \)?

(3 points)

Exercise 3.4.  (a) Consider the Fractional Multi Knapsack Problem:
Given natural numbers \( n, m \in \mathbb{N} \) and \( w_i, c_{ij} \in \mathbb{N} \) as well as \( W_j \in \mathbb{N} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), find \( x_{ij} \) satisfying \( \sum_{j=1}^{m} x_{ij} = 1 \) for all \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} x_{ij} w_i \leq W_j \) for all \( 1 \leq j \leq m \) such that \( \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij} \) is minimum.

State a polynomial-time combinatorial algorithm for this problem.
(Do not use that a linear program can be solved in polynomial time.)

(b) Can we solve the integral Multi Knapsack Problem in pseudopolynomial time if \( m \) is fixed?

(6 points)

Deadline: Thursday, May 3rd, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ss18/appr_ss18_ex.html

In case of any questions feel free to contact me at traub@or.uni-bonn.de.