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Exercise Set 2

Exercise 2.1. The k-CENTER PROBLEM is defined as follows: given an undirected graph G, weights $c : E(G) \to \mathbb{R}_+$, and a number $k \in \mathbb{N}$ with $k \leq |V(G)|$, find a set $X \subseteq V(G)$ of cardinality k such that

$$\max_{v \in V(G)} \min_{x \in X} \mathrm{dist}(v, x)$$

is minimum. As usual we denote the optimum value by $\mathrm{OPT}(G, c, k)$.

(a) Let S be a maximal stable set in $(V(G), \{\{v, w\} : \mathrm{dist}(v, w) \leq 2R\})$. Show that then $\mathrm{OPT}(G, c, |S| - 1) > R$.

(b) Use (a) to describe a 2-factor approximation algorithm for the k-CENTER PROBLEM.

(c) Prove that it is NP-hard to obtain an r-approximation for the k-CENTER PROBLEM for any $r < 2$.

Hint: Use a reduction from the VERTEX COVER PROBLEM.

(7 points)

Exercise 2.2. Consider the standard IP formulation of the MINIMIM WEIGHT SET COVER PROBLEM, and its LP-relaxation

$$\min \left\{ cx \ : \ \sum_{S \in \mathcal{S} : e \in S} x_S \geq 1 \text{ for all } e \in U, \ x_S \geq 0 \text{ for all } S \in \mathcal{S} \right\}.$$

Consider the algorithm that picks all sets associated with non-zero values in an optimum solution to the LP-relaxation. Show that this algorithm achieves an approximation guarantee of p if each element $e \in U$ is contained in at most p sets.

(3 points)

Exercise 2.3. MAXIMUM COVERAGE is the following problem. Given a set U of n elements, a collection $\mathcal{S}$ of subsets of U and an integer k, pick sets $S_1, \ldots, S_k \in \mathcal{S}$ maximizing $\left| \bigcup_{i=1}^{k} S_i \right|$. Consider the greedy algorithm, of iteratively picking the set that contains the maximum number of elements that were not already covered before. (Iterate until k sets are picked.) Show that this algorithm achieves an approximation ratio of

$$\left(1-\left(1-\frac{1}{k}\right)^k\right)^{-1} < \left(1-\frac{1}{e}\right)^{-1} = 1+\frac{1}{e-1}.$$

(5 points)

Exercise 2.4. An instance of MAX-SAT is called k-satisfiable if any k of its clauses can be satisfied simultaneously. Give a polynomial-time algorithm that computes for every 2-satisfiable instance a truth assignment which satisfies at least a $\frac{\sqrt{5}-1}{2}$-fraction of the clauses.

Hint: Some variables occur in one-element clauses (w.l.o.g. all one-element clauses are positive), set them *true* with probability a (for some constant $a \in [0,1]$), and set the other variables *true* with probability $\frac{1}{2}$. Choose a appropriately and derandomize this algorithm.

(5 points)

Deadline: Tuesday, April 24$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss18/appr_ss18_ex.html`

In case of any questions feel free to contact me at traub@or.uni-bonn.de.