Exercise 10.1:
In this exercise we consider variants of the contraction lemma.

(i) Show that the “vertex version” of the contraction lemma is wrong: Define a complete graph whose edge lengths fulfill the triangle inequality and vertex sets $A$, $B$, and $C$ such that

$$0 < \text{mst}(A) - \text{mst}(A \cup C) < \text{mst}(A \cup B) - \text{mst}(A \cup B \cup C).$$

Here, $\text{mst}(X)$ for a vertex set $X$ denotes the length of a minimum spanning tree in the graph induced by $X$.

(ii) Show that the contraction lemma still holds if one adds edges between terminals whose lengths are larger than 0. (“Adding” an edge means that if there already is an edge, a parallel edge is inserted.)

(4+4 points)

Exercise 10.2:
Consider an instance $G = (V, E)$ of the Steiner Tree Problem with terminal set $R$ and edge lengths $c: E \rightarrow \mathbb{R}_+$. Denote the full components of an optimum $k$-Steiner tree $\text{SMT}_k(R)$ with $T^*_1, \ldots, T^*_k$.

(i) Suppose that $V \setminus R$ forms a stable set. Show that

$$\text{mst}(R) \leq 2 \cdot (\text{smt}_k(R) - \text{loss}(T^*_1, \ldots, T^*_k)).$$

(ii) Suppose that all shortest paths between any two vertices in $G$ have length 1 or 2. Show that

$$\text{mst}(R) \leq 2 \cdot (\text{smt}_k(R) - \text{loss}(T^*_1, \ldots, T^*_k)).$$

(iii) Show that the Loss Contraction Algorithm achieves an approximation ratio of $1.279 \cdot r_k$ in both cases.

(2+2+2 points)

Exercise 10.3:
Show that the approximation ratio of the Relative Greedy algorithm is not better than $\frac{41}{30}$. Hint: turn page

(4 points)

Deadline: Thursday, July 6th, before the lecture.
Hint for Exercise 10.3:

Construct instances $G_k$ with
- $4k$ terminals $w_1, \ldots, w_{4k}$,
- three special vertices $T_a, T_b, T_c$,
- $2k$ vertices $y_1, \ldots, y_{2k}$,
- $k$ vertices $z_1, \ldots, z_k$.

Insert edges as follows:
- each terminal $w_{4i-3}$ is connected to $T_a$ by an edge of length $x$, where $2 < x < 2.5$
- each terminal $w_{4i}$ and $w_1$ is connected to $T_b$ by an edge of length $x$
- each $y_i$ is connected to $w_{2i-1}$ and $w_{2i}$ by an edge of length 1
- each $z_i$ is connected to $y_{2i-1}$ and $y_{2i}$ by an edge of length $\frac{1}{2}$
- each $z_i$ is connected to $T_c$ by an edge of length 1
- each terminal $w_{4i-3}$ is connected to $w_{4i-2}$ by an edge of length 2
- each terminal $w_{4i-2}$ is connected to $w_{4i-1}$ by an edge of length 3
- each terminal $w_{4i-1}$ is connected to $w_{4i}$ by an edge of length 2
- each terminal $w_{4i}$ is connected to $w_{4i+1}$ by an edge of length $2x$.

The graph $G_4$ looks as follows: